Problem Set 5:
Government Debt, Default, and the Business Cycle

Exercise 5.1: Strategic default

Consider the government of a small open economy that issues public debt in the form of one-period discount bonds (that means it issues debt of amount b at price q in the first period and pays back b in the second period to the international investors that bought the government debt in the first period). The government has a planning horizon of two periods and is initially in a recession, such that the first period income, $y_1 < y_2$, is smaller than the one of the second period, y_2. In the second period the government can strategically decide to default on the outstanding government debt, b, but suffers a stochastic default cost ϕ if it does so. The benevolent government's objective function is to maximize lifetime utility

$$U = \log(c_1) + \beta \mathrm{E}\left[\log(c_2) - \phi \mathcal{I}\right], \quad 0 < \beta < 1,$$

subject to the constraints

$$\begin{aligned} c_1 &\leq qb + y_1 \\ c_2(\mathcal{I}) &\leq y_2 - b(1 - \mathcal{I}), \end{aligned}$$

where $\mathcal{I}$ is a default indicator, $\mathrm{E}[\cdot]$ is the expectation operator with respect to the default cost ϕ. The cost ϕ is drawn from a uniform distribution with support $[0, \phi_{max}]$, implying that the density function of the default cost is given by

$$f(\phi) = \frac{1}{\phi_{max}}, \quad \phi \in [0, \phi_{max}],$$

and the expectation of a function $g(\phi)$ is given by

$$\mathrm{E}\left[g(\phi)\right] = \int_0^{\phi_{max}} g(\phi) f(\phi) d\phi.$$

(a) We solve this model by backward induction, thus let's first consider the second period. Assume that the realization of the default cost ϕ is known when the government decides over default. The government's indirect utility (or the value) in the second period in case of default ($\mathcal{I} = 1$) is given by

$$V^d(\phi) = \log(y_2) - \phi,$$

while the value of compliance (repayment of the debt) is

$$V^c(b) = \log(y_2 - b).$$

Characterize the threshold realization of the default cost shock, $\Phi(b)$, that makes the government indifferent between compliance and default. What is the governments optimal choice if $\phi < \Phi(b)$? Is the threshold increasing or decreasing in the outstanding debt level b?

(b) What is the lowest debt level b_{max} (out of high debt levels) such that the government will always default? Would any investors ever buy public debt from this government over and above b_{max}?

(c) Now, let's go back to the first period. Compute the probability of a default in the second period

$$\pi(b) \equiv \text{Prob}(\phi < \Phi(b)) \equiv \int_0^{\Phi(b)} \frac{1}{\phi_{max}} d\phi,$$

for a given debt level, $0 \leq b \leq b_{max}$.

(d) Given that the risk-neutral international investors can alternatively invest in a risk-free bond with return R, what will be the rational equilibrium price of the public debt, $q(b)$? Compute the price of debt at $b = 0$ and $b = b_{max}$. (Hint: the risk-neutral international investors have to be indifferent between investing in the alternative with return R and the expected return of the government's discount bond.)

(e) Given the default threshold $\Phi(b)$, the government chooses the debt level, b, to maximize the utility

$$\begin{aligned} U &= \log\left(q(b)b + y_1\right) + \beta \text{E}\left[\log\left(y_2 - b(1-\mathcal{I})\right) - \phi\mathcal{I}\right] \\ &= \log\left(q(b)b + y_1\right) + \beta\left[\int_{\Phi(b)}^{\phi_{max}} \log(y_2 - b) f(\phi) d\phi + \int_0^{\Phi(b)} \left(\log(y_2) - \phi\right) f(\phi) d\phi\right] \\ &= \log\left(q(b)b + y_1\right) + \beta\left[\left(1 - \pi(b)\right)\log(y_2 - b) + \int_0^{\Phi(b)} \left(\log(y_2) - \phi\right) \frac{1}{\phi_{max}} d\phi\right]. \end{aligned}$$

For simplicity, assume that the government takes the default threshold $\Phi(b)$ (and therefore also the default probability $\pi(b)$, and the price $q(b)$) as given such that you can ignore their derivatives with respect to b. Derive the first-order optimality condition with respect to b, and reformulate it in terms of consumption growth. Give an interpretation of the resulting Euler equation.

(f) Assume that $\beta R = 1$. Sketch the optimal consumption path of the government for a given debt level b. Consider two different paths with default cost realization, (i) $\phi > \Phi(b)$, and (ii) $\phi < \Phi(b)$.

Exercise 5.2: Business cycle facts for mainland Norway

In this problem set we establish some business cycle facts about mainland Norway and compare them to the ones found in Hansen (1985, Table 1)[1] for the United States. The data analysis and the questions are cooked up such that you can solve them with the program MS Excel which you can access through the Program Kiosk[2] of the University of Oslo.

[1] Hansen, G., "Indivisible Labor and the Business Cycle," Journal of Monetary Economics, 16 (1985), 281-308.

[2] http://www.uio.no/english/services/it/computer/software/servers/kiosk

(a) Visit the website of Statistics Norway (SSB, www.ssb.no) and download (search for the table number indicated below) aggregate quarterly data for mainland Norway on:

- Statbank, Table 09190 (current prices, seasonally adjusted; 1995K1-2014K2): Gross domestic product Mainland Norway, market values (GDP).
- Statbank, Table 09175 (mainland Norway; 1995K1-2014K2): Compensation of employees (Wages); Employed persons Employees and self-employed, seasonally adjusted (Employment); Total hours worked for employees and self-employed, seasonally adjusted (TotalHours).

The date 1995K1 denotes the first quarter of the year1995 and 2014K2 the second quarter of the year 2014. After extracting the data from the SSB Statbank, manually rotate the tables (klick the circular arrow in the header) such that each time series variable appears as a column vector (put the time dimension in the stub and the variable in the head). Save the tables in MS Excel format and merge the two data files such that you end up with the raw data of the following structure:

Date	GDP	Wages	Employment	TotalHours
1995K1	199'477	104'326	2043,6	756,2
⋮	⋮	⋮	⋮	⋮
2014K2	607'788	335'304	2640,6	934,2

(b) Create two new time series, Productivity = GDP/TotalHours and Hours = TotalHours/Employment by adding two new columns to your data set.

(c) Let the observation of time-series $x \in \{\text{GDP}, \text{Wages}, ..., \text{Hours}\}$ at date t be denoted by x_t. Normalize the gathered macro data by dividing each time series, x_t, by its empirical mean and taking the natural logarithm (you can interpret $\hat{x}_t$ as the approximate percentage deviation from the time series mean)

$$\hat{x}_t \equiv \ln\left(\frac{x_t}{T^{-1}\sum_{t=1}^{T} x_t}\right),$$

where T corresponds to the total number of observations within each time series. (Hint: you could do this normalization in a new Excel worksheet, so you can keep the original structure of the data). Plot $\hat{x}_t$ over time for each variable, do the normalized time-series look stationary or do they have a trend? And if so, what will drive the trend?

Solution:

See Exercise52.xlsx. The general picture is that all variables except Hours seem to have a trend. TotalHours worked and employment look more stationary than the other series, but population growth (leading to growth in the labor force), immigration, and probably the increase in female labor force participation contribute to the slight trend in the series. Technology growth accounts for the remaining trend components in GDP and wages, for example.

(d) Separate the trend component of each normalized time series from the cyclical component

$$\hat{x}_t^{trend} = \hat{x}_t - \hat{x}_t^{cycle},$$

using a Hodrick-Prescott Filter[3] with smoothing parameter $\lambda = 1600$.

(e) Compute the cyclical component of each series

$$\hat{x}_t^{cycle} = \hat{x}_t - \hat{x}_t^{trend}$$

and calculate the standard deviation and (this is optional, you will have to install the Data Analysis add-in of Excel to do this) the cross-correlation of each series relative to GDP. Do you find similar results as in Hansen (1985, Table 1), for example that total hours worked are more volatile than labor productivity? What could be

Series	Std	Corrcoeff
GDP	1.76	1.00
TotalHours	1.66	0.76
Productivity	1.18	0.42

Table 1: Hansen (1985, Table 1)

the reason for this business cycle fact?

Solution:

The reason for this business cycle fact could be the following (see Hansen (1985) for more on this): In the baseline RBC model where all households participate in the labor market and only decide how much to work (the intensive margin of labor supply, labor is perfectly divisible), hours worked should approximately have the same volatility as the labor productivity. However, in the data (and that holds for the U.S. and for mainland Norway), hours worked are more volatile than productivity. What could explain this is the extensive margin of labor supply (the decision whether to participate in the labor market) which becomes important when workers are not completely flexible to adjust the hours worked (indivisible labor units). Namely, there might only be jobs available where you can work a fixed amount of hours (part time, or full time), such that people not only adjust their hours in response to productivity shocks, but possible get out of the labor force completely which could cause the excess fluctuations of the total hours worked observed in the data. Hansen's RBC model with indivisible labor tries to address this issue.

[3]Go to https://wwz.unibas.ch/personendaten/yvanlengwilerunibasch/eigene-seiten/software and download the Hodrick-Prescott filter add-in for Excel and read the explanation and example file on how to install and use the add-in. For some reason, in Excel 2010 the command will be =HP(data;lambda) and not =HP(data,lambda), so you have to separate with a semicolon and not a comma.