Ramsey and Diamond Type Models for Small Open Economies
A short lecture note for ECON 4310

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1 Introduction

The textbook versions of the *Ramsey* and *Diamond* models assume a closed economy. The present lecture note adapts the same two frameworks to a small open economy that can borrow and lend in international capital markets. For those who are interested in digging deeper, more elaborate models can be found in Obstfeld and Rogoff (1996).

The main distinction between the Ramsey and Diamond models is that in the former the decision making is *"dynastic"*, meaning that today's decision makers incorporate the utility functions of their descendants in their own utility functions, while in the Diamond model decisions are made on a more individual basis. The present note first discusses the dynastic and then the individual model of the small open economy. Then the latter is applied to questions on fiscal policy, government debt, pension systems, resource rents and sovereign wealth funds. Throughout we shall be concerned with the determination of the net foreign debt. First, however, we shall have a look at the production side of the economy, which is common to both approaches.

Throughout we assume that there is a single commodity which can be exported and imported freely and used for both consumption and investment in productive capital. It is traded in competitive markets. We also assume that all agents in the economy can borrow and lend freely in international capital markets at a given real interest rate r. In these areas the economy is extremely open, but there is no international labor mobility.

2 The choice of capital intensity and the determination of the real wage

In the closed economy aggregate saving is always equal to aggregate real investment. The possibility to borrow and lend abroad means that this equality does not need to hold. Given our assumptions, the capital intensity of the economy and the level of investment in real capital will be determined independently of domestic savings.

We assume a macro production function with standard properties including constant returns to scale:

$$Y_t = F(K_t, A_t L_t) \tag{1}$$

where Y_t is output, K_t capital input, L_t labor input and A_t a labor-augmenting productivity factor. Productivity and labor inputs are assumed to grow exponentially:

$$A_t = A_0(1+g)^t \tag{2}$$

$$L_t = L_0(1+n)^t \tag{3}$$

Here g is the rate of labor-augmenting technical progress and n is the rate of population growth.

As for the closed economy, it is useful to work with the model in intensive form, which means measuring everything relative to the number of efficiency units of labor. Hence, we define

$$y_t = Y_t/A_t L_t, \qquad k_t = K_t/A_t L_t \tag{4}$$

y_t and k_t are respectively output and capital per efficiency unit of labor. Because F is homogeneous of degree 1, we can write the production function in intensive form as

$$y_t = f(k_t) = F(k_t, 1) \tag{5}$$

Competitive equilibrium requires equality between the marginal productivities of capital and labor and, respectively, the real interest rate and the wage rate. In intensive form these well-known conditions are written:

$$f'(k_t) = r_t \tag{6}$$

and

$$f(k_t) - k_t f'(k_t) = w_t \tag{7}$$

where w_t is the real wage per efficiency unit.

The crucial point, is that the real interest rate r_t is now given from world financial markets and unaffected by the decisions made in the small open economy. This means that the marginal productivity condition (6) alone determines the capital intensity k_t. *The capital stock is independent of domestic savings.* Furthermore, after k_t has been determined by r_t in (6), w_t follows from the other marginal productivity condition (7). Hence, the real wage, w_t, is also independent of domestic savings.

Since the capital stock is independent of domestic savings, the same is obviously also true for real investment. In order to derive an explicit investment equation on intensive form, we can (disregarding depreciation) start from the definitional equation

$$I_t = K_{t+1} - K_t = L_{t+1}A_{t+1}k_{t+1} - L_tA_tk_t \tag{8}$$

where I_t is real investment. Dividing by A_tL_t we get

$$i_t = (1+n)(1+g)k_{t+1} - k_t = (1+\gamma)k_{t+1} - k_t \tag{9}$$

where $\gamma = n + g + ng$ is the natural growth rate. γk_t is the level of investment that is needed in order to keep the capital intensity constant when the effective labor supply grows at the rate γ.

In the sequel we focus on the case where r_t is constant, equal to r. Then k_t is also constant, equal to k. In this case

$$i = \gamma k \tag{10}$$

This we know as one of the conditions for balanced growth. Here it holds every period as long as the international interest rate is constant.

The main lesson from this section is that international capital mobility makes it possible to separate the determination of the level of investment from the determination of the level of savings. Factor prices no longer depend on the domestic savings behavior. The crucial assumptions behind these results are that the country can borrow and lend freely at a given international interest rate and that capital goods can be bought and sold freely in international markets with no transaction costs.

3   The foreign debt and the current account

The converse of the result that the saving decisions of the local consumers do not affect the capital stock, is that these decisions are crucial in determining the net foreign asset position of the economy.

Suppose the total assets of the consumers at the end of period t-1 are a_t (measured by efficiency units of labor). Consumers can invest in either foreign bonds $b_{f,t}$, government bonds, $b_{g,t}$, or bonds issued by firms to finance their real capital. Hence,

$$a_t = b_{g,t} + b_{f,t} + k_t. \tag{11}$$

k_t is determined by firms as described above, a_t by domestic consumers and $b_{g,t}$ by the government. When a_t exceeds $b_{g,t} + k_t$, the remainder has to be invested abroad by buying foreign bonds. If a_t is less than $b_{g,t} + k_t$, the country has to borrow abroad. Then $b_{f,t}$ is negative and represents the foreign debt.

By definition, the current account surplus is equal to the difference between national saving and real investment, or what amounts to the same, the difference between national income (output plus interest on the foreign assets) on the one hand and consumption and investment on the other. When there are no capital gains or losses, the increase from one period to the next in the net foreign assets is equal to the current account surplus. This will be the case in the models below.

In a steady state, where $b_{f,t}$ is constant equal to $\bar{b}_f$, the current account surplus has to be big enough to make the level of foreign assets grow at the rate on natural growth. This means that in steady state the current account surplus is $\gamma \bar{b}_f$. If $\bar{b}_f < 0$, it means that there is a net foreign debt and the current account deficit has to be large enough that the net foreign debt grows at the rate of natural growth.

With these definitions in place, we are ready to study what happens with two different forms of savings behavior.

4   The dynastic model

We shall see that this model yields some disturbing results. Apparently small open economies either end up accumulating endless amounts of debt or enormous wealth.

Consumers maximize

$$U_0 = \sum_{t=0}^{\infty} \beta^t u(C_t) L_t \tag{12}$$

where C_t is consumption *per capita* and $0 < \beta < 1$ is the subjective discount factor. It is related to the subjective discount rate ρ by $\beta = 1/(1+\rho)$. The instantaneous utility function u has standard properties.

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In each period the consumer has a choice between consuming now (C_t) or buying assets to be held until next period, $\tilde{A}_{t+1}$. The amount available to spend is the sum of labor income, $W_t L_t = w A_t L_t$, and the assets carried over from last period plus interest received on them, $(1+r)\tilde{A}_t$. Hence, the period by period budget equation is

$$\tilde{A}_{t+1} = (1+r)\tilde{A}_t + W_t L_t - C_t L_t, \qquad t = 0, 1, 2, \dots \tag{13}$$

As usual we require that the no-Ponzi-game condition is satisfied:

$$\lim_{t\to\infty}(1+r)^{-t}\tilde{A}_{t+1} \geq 0 \tag{14}$$

When the present value of all wage income from now to infinity is finite, the no-Ponzi-game condition implies that the present value of the consumption path from now to infinity should not exceed the initial wealth plus the present value of wage income:

$$\sum_{t=0}^{\infty}(1+r)^{-t}C_t L_t \leq (1+r)\tilde{A}_0 + \sum_{t=0}^{\infty} W_t L_t (1+r)^{-t} \tag{15}$$

Since $W_t L_t$ grows at the rate $(1+g)(1+n)$ the present value of the stream of wages is finite if, and only if, $1+r > (1+g)(1+n)$, or in words, if, and only if, the real interest rate exceeds the natural growth rate. If this is not the case, the country has infinite wealth and no maximum exists for the utility function. Infinite wealth is also inconsistent with the small country assumption. Hence, in the sequel we assume $1+r > (1+g)(1+n)$, or $r > \gamma = g + n + ng$.

Utility maximization gives the usual first-order conditions (Euler equations):

$$u'(C_t) = \beta(1+r)u'(C_{t+1}) \tag{16}$$

Consumption per efficiency unit of labor is by definition $c_t = C_t/A_t$. Hence, we can also write the consumption Euler equation as

$$u'(c_t A_t) = \beta(1+r)u'(c_{t+1}A_{t+1}), \qquad t = 0, 1, 2, \dots \tag{17}$$

The budget equation (13) can be written in terms of c_t, a_t and w_t as:

$$(1+\gamma)a_{t+1} = (1+r)a_t + w_t - c_t \tag{18}$$

From the first-order conditions and the budget constraint we can in principle derive the levels of consumption in each period.

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A balanced growth path is one where $c_{t+1} = c_t = \bar{c}$ for all t. In the closed-economy Ramsey model a balanced growth path exists only for a limited class of utility functions, those that are homothetic. The CRRA utility function is the most important practical example of this class. However, *even with CRRA utility a balanced growth path usually does not exist for the open economy.* With the CRRA utility function

$$u(C) = (1/(1-\theta))C^{1-\theta}, \qquad \sigma = 1/\theta > 0$$

the first order condition (16) specializes to

$$(c_t A_t)^{-\theta} = \beta(1+r)(c_{t+1} A_{t+1})^{-\theta}$$

or

$$c_{t+1} = c_t \frac{[\beta(1+r)]^\sigma}{1+g} \tag{19}$$

From this we see that $c_{t+1} = c_t$ requires

$$[\beta(1+r)]^\sigma = 1+g \iff \frac{1+r}{1+\rho} = (1+g)^{1/\sigma}$$

or

$$r = (1+\rho)(1+g)^{1/\sigma} - 1 = r_c \tag{20}$$

Consumption per capita grows at the same rate as productivity only if the interest rate is at the critical level r_c defined by (20).

Note that the critical interest rate is the same as the level of the interest rate along a balanced growth path in the corresponding closed economy model. There the interest rate adapts endogenously to satisfy condition (20). For the small open economy, however, the interest rate is exogenous . Hence, it seems that in the small open economy balanced growth can only occur by accident. Except in a knife-edge case consumption will either grow faster or slower than productivity forever.

Still much of the literature assumes that (20) holds also for the small open economy. If a justification is given, it is usually that consumers across the world have the same preferences (same ρ and σ) and that productivity growth rates are the same everywhere. Then we can appeal to the closed economy model for the determination of the world interest rate and there (20) holds along a balanced growth path. In this case, if the rest of the world is on a balanced growth path, our small country will also be on a balanced growth path.

The implications of the model for current accounts and foreign debts differ depending on whether we assume a balanced growth path or not.

Case 1. $r = r_c$. **A balanced growth path exists** In this case the interest rate in every period is at the level that makes consumers choose a constant consumption per efficiency unit. In other words $c_t = \bar{c}$, where $\bar{c}$ is the steady-state level of consumption. By setting $a_{t+1} = a_t = \bar{a}$ and $c_t = \bar{c}$ in the budget equation (18), we find that in steady state

$$\bar{c} = w + (r - \gamma)\bar{a} \qquad (21)$$

or consumption expenditure is equal to wage income plus interest income in excess of what needs to be saved in order to keep net assets per efficiency unit of labor constant. Since $c_t = \bar{c}$ from period 0 onwards, we can use (21) to substitute for c_t in the present value budget constraint (15). We will then find that the constraint is satisfied with equality when $\bar{a} = a_0$, that is when consumers save exactly the amount that is needed to keep their asset holdings per efficiency unit of labor at the level they inherited from the past. Intuitively, keeping c_t constant when w is constant requires that a_t is kept constant in order that interest income can continue to pay for the share of consumption, $\bar{c} - w$, that cannot be paid for by wages. Hence, the economy follows the steady state from period 0.

With a_t and k_t constant the country's net foreign assets, $b_f = a - k$, are also constant.

Balanced growth requires that a country with positive net foreign assets keeps a positive current account surplus equal to γb_f forever in order to make its net foreign assets grow at the same rate as the rest of the economy. Conversely, countries with a net foreign debt run a perpetual current account deficit.

It may seem odd that a country continues to have a current account surplus forever. However, since as discussed above, the model is meaningful only if $r > \gamma$, the country consumes a part $r - \gamma$, of the interest income from its net foreign asset. The country's current account surplus is less than its surplus on the interest account, which means that it has a trade deficit. It gets to use more goods than it produces, and in increasing amounts as the economy grows. Similarly, countries with a current account deficit has to have a trade surplus when growth is balanced. They have to send goods abroad.

Case 2. $r \neq r_c$. **No balanced growth path exists** If the real interest rate is above the critical level r_c, consumption will grow faster than the natural growth rate forever. This means that the ratio between consumption and output goes to infinity as time goes to infinity. The only way this can be consistent with the budget constraint is if consumption starts low and the country always runs a current account surplus accumulating more and more claims on foreigners. Thus, the ratio between the country's net foreign assets and output also tend to infinity. Steadily increasing interest income from abroad can then pay for a steadily increasing trade deficit, and still some revenue is left over to be invested in more foreign bonds.

If the interest rate is below the critical level, we get the opposite results: The ratio of consumption to output tends to zero, and the net foreign debt tends to infinity.

Countries with rapid productivity growth (high g), have high critical interest rates. Hence, these countries would tend to be among those that have current account deficits and rapidly increasing foreign debt.[1]

However, with asset or debt ratios going to infinity the countries cease to be small. Hence, we need to look at the world economy and endogenize the real interest rate. In this case it can be shown that in the long run the world economy will tend towards a state where all wealth and all consumption is concentrated in the country with the lowest critical interest rate. We shall not pursue this issue in more detail in here. The extreme conclusions that we get when the economy consists of agents with different preferences throw some doubt on the descriptive value of the model for private sector behavior. As a normative device it may still give some direction. One conclusion is that if natural growth rates and the degree of patience are not very different from the rest of the world, it may be a good idea to save so as to maintain the asset to output ratio (at least if one has utilitarian preferences). High growth countries may want to borrow from abroad in order in order to give some of the expected benefits of future growth to present generations. Countries with low growth may find that the interest costs involved are too heavy to justify a redistribution to early generations that are almost as well off as those who come after them.

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[1]This seems to be contradicted by the experiences of Japan and China in the periods when they had the fastest productivity growth.

5 The individual model

The utility function of the generation born at t is

$$U = u(C_{y,t}) + \beta u(C_{o,t+1}) \tag{22}$$

The budget constraint is

$$C_{y,t} + (1+r)^{-1} C_{o,t+1} = W_t \tag{23}$$

where W_t is the wage earned by generation t when young. Utility maximization yields the first order condition

$$u'(C_{y,t}) = \beta(1+r)u'(C_{o,t+1}) \tag{24}$$

With log utility $u(C) = \ln C$ the first order condition specializes to

$$1/C_{y,t} = \beta(1+r)/C_{o,t+1}$$

or

$$C_{o,t+1} = \beta(1+r)C_{y,t}$$

When this is inserted in the budget equation and solved we get

$$C_{y,t} = \frac{W_t}{1+\beta}, \qquad C_{o,t+1} = \frac{\beta(1+r)W_t}{1+\beta} \tag{25}$$

Individual saving when young is

$$S_{y,t} = W_t - C_{y,t} = \frac{\beta}{(1+\beta)} W_t \tag{26}$$

Hence, the savings rate of the young is

$$\sigma = \frac{\beta}{1+\beta} = \frac{1}{2+\rho} \tag{27}$$

Saving when old is the negative of saving when young:

$$S_{o,t+1} = -S_{y,t}$$

The young save and the old dissave. Since the sum of savings over the individual life-cycle is zero, aggregate savings are positive only if the young are richer or more numerous than the old.

We shall look more closely at the relationship between savings and growth, but first we need some definitions.

Formally total savings in period t are

$$S_t = L_t S_{y,t} + L_{t-1} S_{o,t} \tag{28}$$

where L_t is the size of the young generation at t. The total financial assets of households at the end of period t is

$$\tilde{A}_{t+1} = L_t S_{y,t} \tag{29}$$

The young are the only ones who can carry assets to the next period. Hence, the wealth that is carried over from one period to the next is equal to the savings of the young:

$$\tilde{A}_{t+1} = \sigma w L_t A_t$$

or, if we divide by $A_{t+1} L_{t+1}$,

$$a_{t+1} = \sigma w/(1+n)(1+g) = \bar{a} \qquad t = 0,1,2,\ldots \tag{30}$$

Note that there are no dynamics here. a_t is constant from period 1 onwards. The steady state will always be reached after one period.

As explained in section 2, net foreign assets is determined as a residual between the savings for retirement and the optimal capital stock[2]:

$$b_f = \frac{\sigma w}{(1+\gamma)} - k \tag{31}$$

From period 1 the current account surplus will be equal to γb_f.

One implication is that a country which has positive net foreign assets should continue to have a current account surplus forever. Similarly a country with a net foreign debt should continue to have a current account deficit forever. Unlike in the dynastic model the net foreign debt relative to GDP will never go to infinity. A crucial difference between the dynastic and the individualistic model is that in the former a balanced growth path is compatible with equilibrium only in a knife-edge case, while in the latter an equilibrium with a balanced growth path always exists. The finite lives of the individuals limit how much they can borrow, and how much they can save.

An interesting question is whether debtor countries actually have to hand over goods to creditor countries, i.e. whether debtor counties have to have

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[2]Government debt is set to zero zero.

a trade surplus. Recall that in steady state the current account deficit is $-\gamma\bar{b}_f$. If the current account deficit is smaller in absolute value than the net interest payments to abroad, $-rb_f$, this means that there is a trade surplus. Hence there will be a trade surplus if $r > \gamma$, or, in words, if the real interest rate is greater than the natural growth rate.

If a debtor country faces a real interest rate below its natural growth rate, it gets a free lunch. It can continue forever to use more goods than it produces. However, this is in the model. In the real world interest rates may go up and growth rates may go down. A country with a large debt may then have to pay heavily.

More on savings, investment and the current account

Aggregate saving is equal to the saving of the young minus the dissaving of the old:

$$
\begin{aligned}
S_t &= L_t \sigma W_t - L_{t-1} \sigma W_{t-1} \\
&= L_t \sigma A_t w - L_t (1+n)^{-1} \sigma (1+g)^{-1} A_t w \\
&= \sigma L_t A_t w [(1+n)(1+g) - 1]/[(1+n)(1+g)]
\end{aligned}
$$

This confirms that without growth, $(g = n = 0)$ there will be no net saving. Net assets of the household sector, $\tilde{A}_t$ can still be positive as these are the accumulated result of past savings.

Aggregate savings per efficiency unit of labor is found by dividing with $L_t A_t$ in the last equation:

$$
s = \sigma \frac{(1+n)(1+g) - 1}{(1+n)(1+g)} w = \frac{\beta}{(1+\beta)} \frac{n + g + ng}{(1+n)(1+g)} w = \gamma \bar{a}
$$

The last equality follows from (27) and (30).

The current account surplus is by definition equal to the difference between saving and investment:

$$
ca = s - i = \frac{n + g + ng}{(1+n)(1+g)} \sigma w - (n + g + ng)k = \gamma \bar{a} - \gamma k = \gamma b_f
$$

The last equality follows from (31). Both saving and investment are high in fast-growing countries. Whether the current account is positive or negative depends on whether the young save more or less than what is needed to finance the capital stock for next period.

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6 Government borrowing

Suppose government consumption per efficiency unit of labor is $c_{g,t}$. Suppose also that the government collects a lump sum tax τ_t per efficiency unit from every young worker. If there is a discrepancy between the two, this is covered by government borrowing. The outstanding government debt per efficiency unit of labor is $b_{g,t}$. In sequel we will refer to this as the debt ratio. The debt ratio then grows according to

$$(1+\gamma)b_{g,t+1} = (1+r)b_{g,t} + c_{g,t} - \tau_t \tag{32}$$

$c_{g,t} - \tau_t$ is the *primary deficit* of the government. The behavior of the debt ratio depends on whether r is greater than or less than γ.

Case 1 $r > \gamma$ Suppose a country starts with a positive government debt, $b_{g,t>0}$. Then, if $r > \gamma$ the debt will increase faster than GDP unless there is a sufficiently large primary surplus. This can be seen by dividing equation (32) with $1 + \gamma$ to get

$$b_{g,t+1} = \frac{1+r}{1+\gamma}b_{g,t} + \frac{1}{1+\gamma}(c_{g,t} - \tau_t) \tag{33}$$

When $r > \gamma$, the coefficient in front of $b_{g,t}$ is greater than 1, which means that without a primary surplus, the debt ratio will automatically increase from period to period. Hence, the time path of the debt ratio is governed by an unstable difference equation. When the primary surplus is zero, the evolution of the debt ratio can be seen as a race between two opposing forces. The interest which is added to the debt every period tends to increase the ratio, while the underlying growth of the economy tends to reduce it. When $r > \gamma$, the interest rate wins.

We can find out how large a primary surplus is needed to stop the growth of the debt ratio by setting $b_{g,t+1} = b_{g,t} = b_g$, in (32). This yields

$$\tau - c_g = (r - \gamma)b_g \tag{34}$$

The government has to create a sufficient primary surplus to pay the part of the interest that exceeds the growth rate of the economy. Table 1 shows the primary surpluses needed to stop the growth of the debt ratio at different levels of debt, interest rates and growth rates. The figures can be interpreted as per cents of GDP. If the primary surplus is too small to stop the increase in the debt ratio, the fiscal policy is not sustainable. When

Table 1: Primary surpluses required for stabilizing debt ratios. Per cent of GDP

$r - \gamma$	0.5	1.0	2.0	4.0
b = 50	0.25	0.5	1.0	2.0
b=100	0.50	1.0	2.0	4.0
b= 150	0.75	1.5	3.0	6.0
b=200	1.00	2.0	4.0	8.0

Yearly interest rates and GDP

$r > \gamma$ any policy that stops the increase in the debt ratio at some point satisfies the no-Ponzi-game condition.

In the recent decade interest rates on long-term indexed government bonds issued by US and UK governments have often been 2.0-2.5 per cent. Few, if any investors, would expect to achieve more than four per cent real return on government bonds. Given reasonable assumptions about long-run growth rates, the range for $r - \gamma$ in table 1 should be relevant for many developed countries. Given that in some European countries taxes are close to 50 per cent of GDP, it may seem from table 1 that governments may be able to serve debts that are one or perhaps even two times one year's GDP, although costs in the form of increased tax distortions, less redistribution and reduced government consumption may be formidable. However, if doubts spread that they will actually pay, $r - \gamma$ can soar to levels far outside the range of the table. Serving the debt in full could then be impossible. Important in this connection is the size of the actual primary deficit. If this is e.g. 10 per cent, and stopping the growth of the debt ratio requires a primary surplus of 3 per cent, then a tightening of the budget by 13 per cent of GDP is needed. Politically and legally this can be difficult to achieve fast enough. It is made even more difficult by the fact that the necessary policy in the short run will lead to a fall in GDP that raises the debt ratio and lowers tax revenues.

Case 2: $r < \gamma$ If $r < \gamma$, equation (32) is a stable difference equation in $b_{g,t}$. This means that if we keep the primary surplus constant, the debt ratio will converge to a stationary level. The race between interest and growth is won by growth. Since equation (34) defines the stationary state in (32), we can just solve this to find the stationary level of b_g for a given primary deficit:

$$b_g = \frac{c_g - \tau}{\gamma - r} \qquad (35)$$

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Table 2: Stationary debt ratios for alternative primary deficits. Per cent of GDP

$\gamma - r$	0.5	1.0	2.0	4.0
$c_g - \tau = 1.0$	200	100	50	25
$c_g - \tau = 2.0$	400	200	100	50
$c_g - \tau = 4.0$	800	400	200	100
$c_g - \tau = 8.0$	1600	800	400	200

Year interest rates and GDP

Table 2 shows the debt ratios that will be approached for different levels of the primary deficit and interest rates. The table shows that even small deficits can produce large debt ratios. Ratios can end outside the range in table 1. If the interest rate goes up or the growth rate down, these debt ratios can be difficult to handle and the higher ones are likely to force countries to default. However, if $\gamma - r$ is small, it may take a long time before a country gets near the debt levels in table 2.

The policy we just discussed for $r < \gamma$ implies that the government continues forever to pay all interest with new loans and in addition borrows to finance the primary deficit. Thus, they are running a Ponzi-scheme. The reason governments can do that is that new consumers / lenders are arriving every period in sufficient numbers and with sufficient savings that the old lenders can be repaid fully. Governments are in a unique position to exploit this because they do not have a definite, finite life time and they can collect taxes. The no-Ponzi-game condition is violated, but seems rather too strict in this case. [3]In fact, when $r < \gamma$, it requires that the debt ratio should go to zero or negative numbers as time goes to infinity. In contrast, the no-Ponzi-game condition allows positive debt ratios when the interest rate is higher.

However, it should be remembered that it is the buyers of government bonds who decide how much a country can borrow. They will be concerned with default risk, which is outside the scope of this lecture note. Risk will make the investors want to diversify their bond portfolios. If the country we look at grow faster than the world average, the available funds for investment in that country may grow slower than the country itself.

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[3]An objection is that if the rest of the world grows more slowly, there may be limits to their capacity to increase their holdings of the debts of the fast-growing country, especially if the country is large.

Consequences for the foreign debt   In the extremely open economy the government debt of a small country has no effect on the real interest rate or the capital intensity of the economy. Its main effect will be on the size of the foreign debt. As explained in section 3, the net foreign debt is determined as a residual when the savings of the young are distributed on the different assets available. There are now three of them. Hence,

$$a = b_g + b_f + k \tag{36}$$

The disposable income of the young is now $w - \tau$. Their savings rate out of net income is still σ. Hence,

$$a = \frac{\sigma(w - \tau)}{1 + \gamma} \tag{37}$$

Assume that b_g is kept constant by adjusting the tax rate according to (34). By inserting from (37) and (34) in (36) we get

$$b_f = \frac{\sigma}{1 + \gamma}[w - c_g - (r - \gamma)b_g] - b_g - k \tag{38}$$

We see that when $r > \gamma$ government borrowing reduces net foreign assets (or increases the foreign debt) more than one for one because private savings are also reduced due to higher taxes. Higher government consumption also reduces net foreign assets because the ensuing taxes on the young reduce retirement saving.

If $r < \gamma$, the government can actually reduce taxes permanently if it borrows more. This will benefit all generations of citizens in the form of higher life-time consumption. Net foreign assets would go down, but by less than the increase in government debt. However, this opportunity may tempt many governments to acquire more debt. In turn this will make the world interest rate go up, perhaps to a level where $r > \gamma$.

7   Pension systems and pension reform

In life-cycle models people save for retirement. The need for private saving for retirement will obviously be reduced if a country has a public pension system. Then the young pay a contribution to the system while the old receive a benefit. In a fully *funded* pension system the contributions of the young are set aside in a fund that invests in bonds or other financial assets. Pensions received when old are equal to the contributions paid when young plus the return (expected or realized) on the fund. The opposite case is a

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pure *pay-as-you-go* system. Then there is no fund[4]. The contributions of the young are used to pay the pensions of the old from the previous generations.

Another important distinction is between systems that have defined contributions and those that have defined benefits[5]. In the first case the contributions are fixed and the level of benefits is determined by dividing the amount that resulted from the contributions (including the actual return on the fund, if there is one) on the pensioners. In the second case it is the level of the benefit that is fixed and the contributions are set at the level that is necessary to pay the defined benefit[6]. Actual pension systems are often intermediate cases between the four poles[7]. In the standard life-cycle model with no uncertainty and a constant population growth rate the systems with defined contributions and defined benefits can easily be made equivalent.

Pay-as-you-go   We shall first look at a pay-as-you-go system with defined contributions. Suppose contributions out of wage income are fixed at a rate τ (in per cent of wage income). In period t the young then pay the total amount of $\tau w A_t L_t$. This is shared among the old. Each of them then receives $\tau w A_t L_t / L_{t-1}$. Relative to the income they had when young this is

$$\pi = \frac{\tau w A_t L_t}{w A_{t-1} L_{t-1}} = \tau(1+n)(1+g) = \tau(1+\gamma) \tag{39}$$

π is called the benefit ratio[8]. High growth rates for population and productivity yields a high benefit ratio. Note that implicitly the pension is indexed to the current wage rate. Note also that if we regard the contribution τ as an investment, the rate of return on that investment is the natural growth rate γ.

The first generations of old get pensions even though they did not pay any contributions when young. For all later generations the pension system

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[4]Norwegian terms: Utlikningssystem og fondert system.

[5]Norwegian: Innskottspensjon og ytingspensjon

[6]If a defined benefit system is funded, a mechanism is needed for taking care of cases where the actual return falls short of the expected return.

[7]The Norwegian public pension system is unique in that it is fully integrated in the general budget of the government. Hence, it is not possible to make an economically meaningful distinction between pension contributions and general taxes or between pension funds and general funds.

[8]In Norwegian: pensjonsprosenten.

changes the budget constraint to

$$C_{y,t} + (1+r)^{-1}C_{o,t+1} = (1-\tau)wA_t + (1+r)^{-1}\pi wA_t$$

Net income when young is reduced by the pension contribution. Instead comes the pension income when old. If we exploit that $\pi = \tau(1+n)(1+g)$ we can alternatively write the budget constraint as

$$C_{y,t} + \frac{1}{1+r}C_{o,t+1} = \left[1 + \tau\frac{(1+n)(1+g) - (1+r)}{1+r}\right]wA_t \qquad (40)$$

This reveals that a pay-as-you-go pension increases lifetime income if, and only if,

$$(1+n)(1+g) > (1+r)$$

or the natural growth rate is greater than the interest rate. This is true for all generations that both contribute to the system and receive benefits. When a pay-as-you-go system is first introduced, the generation who is then old gets benefits without having paid a contribution. Hence, if we have dynamic inefficiency $(r < \gamma)$ all generations gain from the introduction of a pay-as-you-go pension system.

This result is analogous to the result that if $r < \gamma$ all generations can gain from a higher government debt. In fact the pay-as you-go system can be viewed as a scheme where the government borrows from the first generation of young to finance pensions for the old, and then rolls over the loan forever by keeping the pension contributions from each generation until they get old, paying an interest rate equal to γ. Some even want to add this implicit debt to the explicit debt in the government accounts[9]. However, there is a crucial difference between the implicit and explicit debt. The former is automatically rolled over since the contributions are dictated by law. The government does not have to go to the market to renew the pension debt.

In the life-cycle model pay-as-you-go pensions have a powerful effect on private savings. This can be illustrated based on an example with log utility. Remember that with log utility consumption when young is a constant share $(1+\rho)/(2+\rho)$ of life-time income. Taking income from (40) we find

$$C_{y,t} = \frac{1+\rho}{2+\rho}\left[1 + \tau\frac{(1+n)(1+g) - (1+r)}{1+r}\right]wA_t$$

which makes the savings rate out of *gross* income when young

$$s(r) = \frac{(1-\tau)wA_t - C_{y,t}}{wA_t} = \frac{1}{2+\rho} - \tau - \left(\frac{1+\rho}{2+\rho}\right)\left(\frac{\gamma - r}{1+r}\right)\tau \qquad (41)$$

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[9]The implicit pension debt at the end of one period b_p is equal to the present value at of the pension payments to be made in the next period; i.e. $b_p = \tau(1+\gamma)w/(1+r)$.

The first term in (41) is the savings rate that applies if there are no public pensions. The second term is the main effect of the pay-as-you-go pension. It reduces savings by about twice as much as a comparable increase in ordinary taxes on the young would have done (exactly twice if $\rho = 0$). In both cases there is less income to save from in the first period, but with a pension there is also less need to save for the second period. The main motive for saving is reduced. The third term in (41) is the effect on savings of the change in life-time income that occurs if $r \neq \gamma$.

The reduction in private saving will show up in the size of the foreign debt. Everything else equal, countries with pay-as-you-go pensions should have more foreign debt.

Funded pensions   As before the contribution paid by a worker who is young in period t is $\tau w A_t$. However, the pension he gets next period is $(1 + r)\tau w A_t$. Taking account of the net income flows that he now receives his budget constraint is then

$$C_{y,t} + (1+r)^{-1}C_{o,t+1} = (1-\tau)wA_t + (1+r)^{-1}(1+r)\tau wA_t = wA_t$$

This shows that the pension system has no effect on the consumer's life-time wealth. His budget constraint is the same as without a pension system. He then chooses the same consumption levels. The saving that is done in the pension fund is equivalent to the saving he would have done himself. His personal savings are reduced by the same amount that is saved in the pension system. The country's total savings will be the same as without a pension system. Hence, the foreign debt is not affected [10].

If $r > \gamma$, the implicit return that people get on their pension contributions are greater in the funded system than in the pay-as-you-go system. This

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[10]One may ask what is then the point of introducing a public pension system. Among the reasons given are: 1) Some people lack the foresight and willpower to save enough for retirement; 2) People do not save enough for retirement because they expect the government to help them out in any case; 3) Adverse selection problems related to longevity insurance. The first two points, if valid, would lead to the conclusion that a funded pension system raises total savings somewhat, and that a pay-as-you-go system has a less negative effect on national savings than the simple life-cycle model predicts. The effect of the third point on total savings is more ambiguous.

has led some people to argue that everybody would be better off if pay-as-you-go systems are replaced by funded systems. This is a misunderstanding. The transition to a funded system requires that somebody pays the pensions of those who earned pension rights in the pay-as-you go system. Suppose generation t is the first generation that is enrolled in the funded system. They should have paid $\tau w A_t$ to the old from the previous system. Suppose the government borrows this amount and rolls over the loan forever paying only the part of interest that exceeds the natural growth rate. Then future generations will need to pay an additional tax equal to $(r - \gamma)\tau w A_t$ to pay for the pension rights that were earned before the reform. This is equal to the difference in return between the two systems. Hence, the apparent gain from the transition to a funded system disappears when we take account of the need to pay for the first generation of old. The only thing that the transition does is to make the implicit pension debt explicit.

8 Petroleum revenues and sovereign wealth funds

Suppose the government has revenues from an exhaustible resource, e.g. petroleum. We can define the petroleum wealth as the expected present value of the resource rent that accrues to the government. This is a measure of the value to the government of the petroleum that is still under ground. Let $a_{p,t}$ be the petroleum wealth at the end of period $t - 1$, and let $y_{p,t}$ be the net flow of resource rents to the government in period t, both measured per efficiency unit of labor. The petroleum wealth, $a_{p,t}$, will then evolve over time according to a law of motion with the same structure as we have seen for other kinds of wealth before:

$$(1 + \gamma)a_{p,t+1} = (1 + r)a_{p,t} - y_{p,t} \tag{42}$$

Without any extraction the wealth would increase with the interest rate from one period to the next just because the time of extraction comes nearer. Extraction reduces the wealth.

Calculating the petroleum rent and its change over time.

If you are not immediately convinced of equation (42), this box contains a proof. The petroleum wealth per efficiency unit of labor that is carried over from period $t-1$ to period t is the present value at $t-1$ of all future petroleum revenues divided by the number of efficiency units in period t:

$$a_{p,t} = \frac{1}{A_t L_t} \sum_{j=0}^{\infty} (1+r)^{-j-1} Y_{p,t+j}$$

The petroleum wealth at the end of period t is then

$$a_{p,t+1} = \frac{1}{A_{t+1} L_{t+1}} \sum_{j=0}^{\infty} (1+r)^{-j-1} Y_{p,t+1+j}$$

If you first split the expression for $a_{p,t}$ in one term for period t and one term for the remaining periods, and then take advantage of the relation between the last term and $a_{p,t+1}$ you get

$$
\begin{aligned}
a_{p,t} &= \frac{1}{A_t L_t} \sum_{j=0}^{\infty} (1+r)^{-j-1} Y_{p,t+j} \\
&= \frac{1}{A_t L_t} (1+r)^{-1} Y_{p,t} + (1+r)^{-1} \frac{1}{A_t L_t} \sum_{j=0}^{\infty} (1+r)^{-j-1} Y_{p,t+1+j} \\
&= (1+r)^{-1} y_{p,t} + (1+r)^{-1} \frac{A_{t+1} L_{t+1}}{A_t L_t} a_{p,t+1}
\end{aligned}
$$

Multiply with $1+r$ on both sides and you get

$$(1+r) a_{p,t} = y_{p,t+1} + (1+\gamma) a_{p,t+1}$$

which is the same as (42).

The net wealth of the government is

$$a_{g,t} = a_{p,t} - b_{g,t}$$

It develops over time according to

$$
\begin{aligned}
(1+\gamma) a_{g,t+1} &= (1+\gamma) a_{p,t+1} - (1+\gamma) b_{g,t+1} \\
&= (1+r) a_{p,t} - y_{p,t} - (1+r) b_{g,t} - c_{g,t} + \tau_t + y_{p,t}
\end{aligned}
$$

Here $\tau_t + y_{tp,t} - c_{g,t}$ is the government's primary surplus as conventionally measured with petroleum rent counted as income. However, the petroleum

rent has an offset earlier in the equation which is the reduction of petroleum wealth. When we consolidate the equation, the petroleum rent drops out and we are left with

$$(1 + \gamma)a_{g,t+1} = (1 + r)a_{g,t} - c_{g,t} + \tau_t \qquad (43)$$

The implication is that we should not consider the petroleum rent as ordinary income. It is better seen as an asset sale. However, the implicit interest income from the petroleum wealth we should consider at par with other income from capital.

Suppose we want to keep taxes and government expenditure per efficiency unit of labor constant. Then in the long run we will also need to keep the net assets of the government constant. If we set $a_{g,t+1} = a_{g,t} = \bar{a}_g$ (43) gives us the condition:

$$c_g - \tau = (r - \gamma)\bar{a}_g \qquad (44)$$

This gives the following rule for budgeting: *Spend the part of the real return on government wealth that exceeds the growth rate of the economy.* If this rule is followed, not only will other assets replace the petroleum wealth as it declines, total assets of the government will also grow with the same rate as GDP (exclusive of oil rents). One argument in favor of this rule is that in practice the government will need to use distortionary taxes. Distortions usually increase more than proportionally with the tax rates. Tax-smoothing, keeping the same tax rate from year to year, is therefore more efficient than varying tax rates. If c_g varies from period to period, tax smoothing means that the primary deficit should vary according to whether current expenditures are above or below the expected long-term average.

In the first years after an oil discovery the interest on the petroleum wealth will often exceed the current oil revenues. Then following the fiscal rule (44) may mean borrowing in order to consume some of the oil revenues in advance. In the next stage oil revenues may be used to pay down debt. The third stage would be to build up assets abroad in what internationally is known as a sovereign wealth fund.

The spending rule (44) gives every generation a share in the oil revenues. However, one can argue that the resulting intergenerational distribution is unfair towards the first generations. Later generations will enjoy a higher living standard due to higher productivity[11]y. Both private and government consumption will be higher. Why shouldn't the first generations get more? What would a utilitarian social planner do? The dynastic model in section

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[11]Or will they suffer from damage from climate change or from changes in the international division of labor?

4 can tell us something about this. The analysis there implies that if the international real interest rate is equal to the critical level

$$r^c = (1 + \rho)(1 + g)^{-1/\sigma}$$

then the social planner will choose a consumption path that increases with the rate of productivity growth, just as with the fiscal rule (44). A path with lower consumption growth, and higher consumption for the first generations will be chosen if $r < r^c$. This choice is more likely the more the social planner discounts the future and the more she dislikes inequality (the lower her σ)[12]. Note though, that if the policy of high consumption early on and low consumption growth is followed for too long, the country may end up being extremely poor relative to the world average.

Other criteria for justice, e.g. Rawl's maximin principle, may lead to consumption levels that start higher even if $r > r^c$. In any case the distributional advantages of alternative paths must be weighed against the efficiency costs of having variable tax rates. One also has to take account of that the expected productivity growth or the expected oil revenues, may not materialize. Strategies that imply very high levels of debt or of assets are more vulnerable to interest rate risk (or rate of return risk).

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[12]Presumably these preferences should also be reflected in her attitudes to redistribution within generations.

The Norwegian Fiscal Rule and Petroleum Fund

Since 1996 Norway has channeled oil revenues into a sovereign wealth fund. Slightly simplified it functions like this: Each year the government's net revenues from the petroleum sector is channeled into the fund. Without oil revenues the government runs a deficit. This is financed by the fund, which passes on the required amount straight to the government. The rest the fund invests in international financial markets.

Since 2001 this set-up has been supplemented by a rule for fiscal policy which says that in normal years the government should keep the budget deficit exclusive of oil revenues equal to four per cent of the value of the petroleum fund. In other words the transfer from the petroleum fund to the government budget should be four per cent of the value of the fund, and when this transfer has been taken account of, the budget should be balanced. Although it has never been stated officially, four per cent per year was probably the real return that was expected from the fund as a long-run average. Compared to the rule suggested in (44) there are two obvious differences:

1. The rule applies to the petroleum fund, not total government wealth. In particular the value of oil that is still in the ground is neglected. This reduces the amount that can be spent. The effect will diminish over time as less oil is left in the ground.

2. The rule says one should spend the whole of the interest income, not just the part that exceeds the natural growth rate. This raises the amount that can be spent. This effect is increasing over time as the fund is filled with more oil revenue.

A stated purpose when the rule was introduced was to make sure that the phasing-in of oil revenues would be gradual. Point 1 above contributes to that. Point 2 means that if economic growth and population growth continues after all oil reserves have been exploited, the petroleum fund will become gradually less important in the national economy. Early generations get more than there is room for if the ratio of assets to income were to be maintained forever, but their living standard is still believed to be lower than that of later generations.

As for tax smoothing, simulations indicate that the Norwegian fiscal rule is not sufficiently strict. Large tax increase or deep cuts in government expenditures will be needed even if the rule is adhered to. This is related to aging of the population and to the maturing of the public pension system. Whether this is the result of a choice between efficiency and equity or just political expediency we leave an open question.