

## Final Exam II

### ECON 4310, Fall 2015

1. Do **not write with pencil**, please use a ball-pen instead.
2. Please answer in **English**. Solutions without traceable outlines, as well as those with unreadable outlines **do not earn points**.
3. Please start a **new page** for **every** short question and for every subquestion of the long questions.

Good Luck!

	Points	Max
Exercise A		60
Exercise B		60
Exercise C		60
$\Sigma$		180

## Exercise A: Short Questions (60 Points)

Answer each of the following short questions on a separate answer sheet by stating as a first answer True/False and then give a short but instructive explanation. You can write, calculate, or draw to explain your answer. You only get points if you have stated the correct True/False *and* provided a correct explanation to the question. We will not assign negative points for incorrect answers.

### Exercise A.1: (10 Points) Finite horizon model of intertemporal consumption

Consider the optimal intertemporal consumption choice of a household in discrete and finite time  $t = 0, 1, \dots, T < \infty$ . The optimal behavior is characterized by the consumption Euler equation

$$\frac{c_{t+1}}{c_t} = [\beta(1 + r - \delta)]^{1/\theta}, \forall t \leq T,$$

and the private budget constraint

$$a_{t+1} + c_t = (1 + r - \delta)a_t + w, \forall t \leq T,$$

taking as given initial and terminal conditions

$$\begin{aligned} a_0 &= 0 \\ a_{T+1} &= 0, \end{aligned}$$

where  $a_t$  denotes asset holdings,  $r - \delta$  is the exogenous interest rate,  $w$  the constant and exogenously given wage income,  $c_t$  the individual consumption of the household,  $\delta \in (0, 1)$  the depreciation rate of physical capital,  $\beta \in (0, 1)$  is the subjective discount factor, and  $1/\theta$  the intertemporal elasticity of substitution.

Suppose that  $\beta(1 + r - \delta) = 1$ . Then this household will first accumulate strictly positive assets for a while and then run down assets over the life-cycle. True or false?

#### **Your Answer:**

True       False:

The parameter restriction  $\beta(1 + r - \delta) = 1$  implies - through the consumption Euler equation - that consumption will be constant over the life-cycle. As the household starts and ends with zero assets, the only way to keep consumption possibilities constant is to keep assets at the zero level and to optimally consume the wage  $w$  in every period.

**Exercise A.2: (10 Points) OLG model, a great famine**

Consider the capital accumulation equation of the overlapping generations model with exogenous technology and population growth.

$$k_{t+1} = \frac{(1-\alpha)\beta}{(1+\beta)(1+g)(1+n)} k_t^\alpha, \quad k_t \equiv K_t/(A_t L_t),$$

where  $K_t$  is the aggregate capital stock,  $A_t$  is the state of technology,  $L_t$  the size of the young cohort,  $\beta \in (0,1)$  the discount factor,  $\alpha \in (0,1)$  the capital income share in the economy, and  $g \geq 0$  and  $n \geq 0$  denote the net growth rate of technology and the cohort size, respectively. The competitive interest rate is given by

$$r_t - \delta = \alpha k_t^{\alpha-1} - \delta.$$

Let the economy be in the stable steady-state with capital per efficiency unit,  $k^* \equiv K_t/(A_t L_t) > 0$ . Now assume that in period  $t_0$ , unexpectedly, a great famine hits this economy such that the current cohort of young is reduced from  $L_{t_0}$  to  $L'_{t_0} < L_{t_0}$ . In response to this unexpected shock the interest rate will jump down on impact and then increase as the economy converges gradually back to the initial steady-state. True or false?

**Your Answer:**True: False: 

The reduction in the size of the current cohort will increase the capital stock per efficiency unit in period  $t_0$  from  $k^* \equiv K_{t_0}/(A_{t_0} L_{t_0})$  to  $K_{t_0}/(A_{t_0} L'_{t_0}) > k^*$ . Thus, the interest rate will jump down in the shock period  $t_0$ . After the shock, since the functional form of the capital accumulation equation per efficiency unit is unchanged, the economy will reduce the capital stock per efficiency unit gradually in the transition back to the initial steady-state level. In the transition the interest rate therefore will gradually increase.

**Exercise A.3: (10 Points) Ramsey model, technology shock**

Consider the dynamic equilibrium equations of the Ramsey model with exogenous growth in technology

$$\frac{c_{t+1}}{c_t} = \left[ \frac{\beta}{1+g} (1 + \alpha k_{t+1}^{\alpha-1} - \delta) \right]^{1/\theta}, \quad c_t \equiv C_t / (A_t L),$$

$$k_{t+1} - k_t = k_t^\alpha - c_t - \delta k_t - g k_{t+1}, \quad k_t \equiv K_t / (A_t L),$$

where  $K_t$  is the aggregate capital stock,  $\alpha k_{t+1}^{\alpha-1} - \delta$  the interest rate,  $g \geq 0$  is the constant net growth rate of technology,

$$A_{t+1} = (1+g)A_t, \quad A_0 > 0,$$

$L = 1$  the constant size of the population,  $C_t$  aggregate consumption,  $\alpha \in (0,1)$  the capital income share in the economy,  $\delta \in (0,1)$  the depreciation rate of physical capital,  $\beta \in (0,1)$  is the subjective discount factor, and  $1/\theta$  the intertemporal elasticity of substitution.

Suppose that the economy is in the steady-state with capital stock per efficiency unit,  $k^* \equiv K_t / (A_t L)$ . Now assume that in period  $t_0$ , the level of technology jumps up from  $A_{t_0}$  to  $A'_{t_0} > A_{t_0}$  due to an unexpected innovation. In response to this temporary one-period shock consumption per efficiency unit,  $c_t$ , will jump down on impact. True or false?

**Your Answer:**

True:

False:

The increase in the level of technology will reduce the capital stock per efficiency unit in period  $t_0$  from  $k^* \equiv K_{t_0} / (A_{t_0} L)$  to  $K_{t_0} / (A'_{t_0} L) < k^*$ . Moreover, since the form of the equilibrium conditions remains unchanged by the shock, starting from period  $t_0 + 1$  the economy will converge back to the initial steady-state on the stable saddle-path (which goes from South-West to North-East of the steady-state in the phase diagram). Thus, consumption per efficiency unit must jump down together with the capital stock per efficiency unit on impact. Intuitively, the economy cuts back in consumption per capita to build up capital.

**Exercise A.4: (10 Points) Ramsey model, Golden Rule capital stock**

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Consider the capital accumulation equation of the Ramsey model with exogenous technology and population growth

$$(1 + g)(1 + n)k_{t+1} = k_t^\alpha - c_t + (1 - \delta)k_t, \quad k_t \equiv K_t/(A_t L_t), \quad c_t \equiv C_t/(A_t L_t),$$

where  $K_t$  is the aggregate capital stock,  $A_t$  is the state of technology,  $L_t$  the size of the population,  $C_t$  aggregate consumption,  $\alpha \in (0, 1)$  the capital income share in the economy,  $\delta \in (0, 1)$  the depreciation rate of physical capital,  $g \geq 0$  denotes the net growth rate of technology, and  $n \geq 0$  is the net growth rate of population. The competitive interest rate is given by

$$r_t - \delta = \alpha k_t^{\alpha-1} - \delta.$$

The Golden Rule capital stock per efficiency unit (the capital stock per efficiency unit that maximizes steady-state consumption per efficiency unit) implies an interest rate of

$$r_{GR} - \delta = (1 + g)(1 + n) - 1.$$

True or false?

**Your Answer:**

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True:

False:

Solve the capital accumulation equation for consumption and impose the steady-state condition,  $k_t = k_{t+1} = k$ . The Golden Rule capital stock is then characterized by

$$k_{GR} = \arg \max_{k \geq 0} k^\alpha - \delta k + [1 - (1 + g)(1 + n)]k,$$

with the associated optimality condition

$$0 = \alpha k_{GR}^{\alpha-1} - \delta + [1 - (1 + g)(1 + n)] \Leftrightarrow r_{GR} - \delta = (1 + g)(1 + n) - 1.$$


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**Exercise A.5: (10 Points) Precautionary savings motive**

Consider the simple two-period real business cycle model discussed in the seminar and in the lecture. With an asset supply of zero,  $w_1 = E[w(s_2)]$ , and an optimal consumption profile,  $c_1 = w_1, c_2(s_2) = w(s_2)$ , the stochastic consumption Euler equation in this model is given by

$$\beta(1 + r_2) = \frac{u'(c_1)}{E[u'(c_2(s_2))]} = \frac{u'(E[w(s_2)])}{E[u'(w(s_2))]}.$$

The stochastic process for the wage in the second period,  $w(s_2)$ , takes the form

$$w(s_2) = \begin{cases} w(s_G) = 1 + \sigma/2, & \text{with prob. } 1/2, \\ w(s_B) = 1 - \sigma/2, & \text{with prob. } 1/2, \end{cases}$$

where  $\sigma \in [0, 2)$  parametrizes the risk in this economy. Assume that the utility function is of the following form

$$u(c) = 1 - \frac{1}{\alpha} e^{-\alpha c}, \quad \alpha > 0.$$

This utility function,  $u(c)$ , implies that there is a precautionary savings motive in this economy. True or false?

**Your Answer:**

True:

False:

The marginal utility function is strictly convex as

$$u'(c) = e^{-\alpha c} > 0$$

$$u''(c) = -\alpha e^{-\alpha c} < 0$$

$$u'''(c) = \alpha^2 e^{-\alpha c} > 0.$$

Thus, there is indeed a precautionary savings motive in this economy.

**Exercise A.6: (10 Points) Precautionary savings and labor supply**

Consider the simple two-period real business cycle model discussed in the seminar and in the lecture. The stochastic consumption Euler equation reads

$$\beta(1 + r_2) = \frac{u'(c_1)}{E[u'(c_2(s_2))]}.$$

Let the asset supply be fixed to zero such that the optimal consumption is  $c_1 = w_1$  and  $c_2 = w(s_2)$ , and let the utility function  $u(c)$  be such that there is a precautionary savings motive in this economy (you don't need to check this).

The stochastic process for the wage in the second period,  $w(s_2)$ , takes the form

$$w(s_2) = \begin{cases} w(s_G) = 1 + \sigma/2, & \text{with prob. } 1/2, \\ w(s_B) = 1 - \sigma/2, & \text{with prob. } 1/2, \end{cases}$$

where  $\sigma \in [0, 2)$  parametrizes the risk in this economy. Moreover, the wage in period 1 corresponds to the expected wage in period 2,  $w_1 = E[w(s_2)]$ .

The equilibrium interest rate  $1 + r_2$  will be equal to  $1/\beta$  in an economy without risk,  $\sigma = 0$ , and must be strictly lower than  $1/\beta$  in an economy with strictly positive risk,  $\sigma = 1$ . True or false?

**Your Answer:**

True:                       False:

Without risk,  $\sigma = 0$ , the wage income is the same in all periods and states,  $w_1 = w(s_2) = 1$ , such that also consumption is the same in all periods and states. The Euler equation then reads

$$\beta(1 + r_2) = \frac{u'(w_1)}{E[u'(w(s_2))]} = 1 \quad \Leftrightarrow \quad 1 + r_1 = 1/\beta.$$

Because there is a precautionary savings motive in the economy, increasing the risk to  $\sigma = 1$  will increase the demand for assets relative to the economy without risk for any given interest rate. However, since assets are available in zero supply, the only way to bring the capital market in the equilibrium is to reduce the interest rate such that the equilibrium demand for assets remains at zero.

## Exercise B: Long Question (60 Points)

### Consumption response to income shocks

Consider a household decision problem under uncertainty, when preferences are linear-quadratic:

$$u(c_t) = c_t - \frac{b}{2}c_t^2, \quad b \geq 0.$$

The household lives from period 0 to period  $T < +\infty$  and discounts the future with factor  $\beta = 1$ . Assume  $1 + r = 1/\beta = 1$ , that is  $r = 0$ . The lifetime budget constraint of the household viewed from period 0 reads

$$(1 + r)a_0 + \sum_{t=0}^T \frac{I_t}{(1 + r)^t} = \sum_{t=0}^T \frac{c_t}{(1 + r)^t}, \quad (1)$$

and viewed from period 1

$$(1 + r)a_1 + \sum_{t=1}^T \frac{I_t}{(1 + r)^t} = \sum_{t=1}^T \frac{c_t}{(1 + r)^t}, \quad (2)$$

where  $I_t$  is income in period  $t$  which is uncertain in any previous period  $s \leq t$  and only learned in period  $t$ . Expected income (or, consumption) in period  $t$  viewed from period  $s \leq t$  equals  $E_s[I_t]$  (or,  $E_s[c_t]$ ), and the Euler equation is given by

$$u'(c_t) = \beta(1 + r)E_t[u'(c_{t+1})], \quad \forall t \leq T - 1. \quad (3)$$

- (a) (10 Points) Under the given assumptions, (i) show that the Euler equation can be written as

$$c_t = E_t[c_{t+1}], \quad \forall t \leq T - 1, \quad (4)$$

and, (ii) give an interpretation (just one sentence!) of Equation (4).

#### **Solution:**

*XX Allocation of points:*

*(i) derivation of the Euler equation (5 points), (ii) correct interpretation of Equation (4) (5 points). XX*

- (i) Since  $\beta(1 + r) = 1$ , and marginal utility is given by

$$u'(c_t) = 1 - bc_t,$$

the Euler equation (3) can be written as

$$1 - bc_t = \beta(1 + r)E_t[1 - bc_{t+1}] \Leftrightarrow c_t = E_t[c_{t+1}].$$

- (ii) The above Euler equation implies that future consumption is expected to be the same as today's consumption. *XX The following step is just for illustration, and*



NOT required in the answer to the question. XX The law of iterated expectations then implies

$$\begin{aligned} E_s [c_t] &= E_s [E_{s+1} [\dots E_{t-2} [E_{t-1} [c_t]] \dots]] \\ &= E_s [E_{s+1} [\dots E_{t-2} [c_{t-1}] \dots]] \\ &= E_s [E_{s+1} [c_{s+2}]] \\ &= E_s [c_{s+1}] = c_s. \end{aligned}$$

- (b) (10 Points) Note that Equation (4) together with the law of iterated expectations implies that

$$E_s [c_t] = c_s, \quad \forall s \leq t.$$

Now, assume that the household expects to not receive any subsidy such that income  $I_t$  is equal to wage income  $w_t$  in any period  $t$ . Take expectations on the lifetime budget constraints and show that consumption in period 0 and 1 are given by

$$c_0 = (T + 1)^{-1} \left( a_0 + \sum_{t=0}^T E_0 [I_t] \right), \quad E_0 [I_t] = E_0 [w_t],$$

and

$$c_1 = T^{-1} \left( a_1 + \sum_{t=1}^T E_1 [I_t] \right), \quad E_1 [I_t] = E_1 [w_t],$$

respectively.

**Solution:**

XX Allocation of points:

(i) taking expectation of the lifetime budget constraint (2 points for each  $s = 0, 1$ ), (ii) solving for consumption (3 points for each  $s = 0, 1$ ). PLEASE, DO NOT AWARD POINTS if consumption is not derived XX

- (i) Take expectations from period  $s = 0, 1$  on both sides of the lifetime budget constraint using the fact that  $E_s [c_t] = c_s$

$$a_s + \sum_{t=s}^T E_s [I_t] = \sum_{t=s}^T E_s [c_t] = (T + 1 - s)c_s,$$

where we have also used the fact that  $r = 0$ . Divide both sides by  $T + 1 - s$  to yield

$$c_s = (T + 1 - s)^{-1} \left( a_s + \sum_{t=s}^T E_s [I_t] \right), \quad s = 0, 1.$$

Finally, expectations income in period  $s$  are given by  $E_s [I_t] = E_s [w_t]$ .

- (c) (10 Points) Now assume that the household knows (already in period 0) that from period 1 to period  $T$  the government pays a subsidy  $S$ , so that income is  $I_t = w_t + S$  from period 1 onwards. Derive again consumption in period 0 and 1.

**Solution:**

*XX Allocation of points:*

*(i) consumption in period 0 (5 points), (ii) consumption in period 1 (5 points). PLEASE ALLOCATE 2 POINTS for (i) and (ii), if the  $g$  is correctly taken into account, but the expectations are wrong, i.e.,  $E_s[I_t]$  instead of  $E_s[w_t]$ . XX*

Simply use the information in the above derived expressions in period 0 consumption

$$\begin{aligned} c_0 &= (T+1)^{-1} \left( a_0 + w_0 + \sum_{t=1}^T (E_0[w_t] + S) \right) \\ &= (T+1)^{-1} \left( a_0 + \sum_{t=0}^T E_0[w_t] \right) + \frac{T}{T+1} S, \end{aligned}$$

and in period 1 consumption is

$$\begin{aligned} c_1 &= T^{-1} \left( a_1 + \sum_{t=1}^T (E_1[w_t] + S) \right) \\ &= T^{-1} \left( a_1 + \sum_{t=1}^T E_1[w_t] \right) + S. \end{aligned}$$

- (d) (10 Points) Assume now that in period 0 the household expects to never receive a subsidy. In period 1 the household is surprised since the government now pays a subsidy  $S$  from period 1 to period  $T$ , so that income  $I_t = w_t + S$  from period 1 to  $T$ . Derive consumption in period 0 and 1.

**Solution:**

*XX Allocation of points:*

*(i) consumption in period 0 (5 points), (ii) consumption in period 1 (5 points). PLEASE ALLOCATE FULL POINTS, if the impact of  $g$  is correctly incorporated, but there is a repeated mistake based on wrong calculations in part (b) or (c). XX*

Consumption in period 0 is the same as in part (b)

$$c_0 = (T+1)^{-1} \left( a_0 + \sum_{t=0}^T E_0[w_t] \right),$$

while consumption in period 1 is the same as in part (c)

$$\begin{aligned} c_1 &= T^{-1} \left( a_1 + \sum_{t=1}^T (E_1[w_t] + S) \right) \\ &= T^{-1} \left( a_1 + \sum_{t=1}^T E_1[w_t] \right) + S. \end{aligned}$$

- (e) (10 Points) Assume now that in period 0 the household expects to never receive a subsidy. In period 1 the household is surprised since the government now pays a subsidy  $S$  but only in period 1. From period 2 to period  $T$  no subsidies are paid. Derive consumption in period 0 and 1.

**Solution:**

*XX Allocation of points:*

*(i) consumption in period 0 (5 points), (ii) consumption in period 1 (5 points). PLEASE ALLOCATE FULL POINTS, if the impact of  $g$  is correctly incorporated, but there is a mistake based on wrong calculations in part (b), (c), or (d). XX*

Consumption in period 0 is the same as in part (b)

$$c_0 = (T + 1)^{-1} \left( a_0 + \sum_{t=0}^T E_0[w_t] \right),$$

while consumption in period 1 is

$$\begin{aligned} c_1 &= T^{-1} \left( a_1 + E_1[w_1] + S + \sum_{t=2}^T E_1[w_t] \right) \\ &= T^{-1} \left( a_1 + \sum_{t=1}^T E_1[w_t] \right) - T^{-1}S. \end{aligned}$$

- (f) (10 Points) Explain your results and the differences in consumption for the different scenarios in parts (c), (d), and (e), relative to part (b).

**Solution:**

*XX Allocation of points:*

*(i) correct explanation of scenario in (c) (4 points), (ii) correct explanation of scenario in (d) (3 points), (iii) correct explanation of scenario in (e) (3 points). XX*

- In part (c), a permanent and expected increase in income gives a permanent (net present-value equivalent) increase in both consumption levels, in period 0 and 1.

- In part (d), a permanent and unexpected increase in income leaves period 0 consumption unchanged, but gives a one-to-one expansion of period 1 consumption.
  - In part (e), a temporary and unexpected increase in income leaves period 0 consumption unchanged, and gives only a small expansion of period 1 consumption, because in lifetime value the one-period increase in the subsidy has less impact than the permanent increase in part (d).
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## Exercise C: Long Question (60 Points)

### Labor Supply

Consider a representative consumer living for two periods, denoted by  $t \in \{1, 2\}$ , in a small open economy. The consumer has preferences over consumption,  $c_t$ , and the hours of labor supplied,  $h_t$ , of the following form

$$U = \log(c_1) - \frac{\phi}{2}(h_1)^2 + \beta \left[ \log(c_2) - \frac{\phi}{2}(h_2)^2 \right],$$

where  $\log$  denotes as always the natural logarithm. The real wage is  $w_1$  in period 1 and  $w_2$  in period 2. The consumer has no capital income in the first period as she starts life without any assets, but the consumer may transfer income between periods (savings) at the exogenous world interest rate  $r$ .

- (a) (15 Points) Set up the optimization problem of this consumer and derive the optimality conditions. Note that consumption, labor supply, and savings are the choice variables of this optimization problem. (hint: you can do the optimization subject to two period-by-period budget constraints or subject to a single net present value budget constraint.)

#### **Solution:**

*XX Allocation of points: (i) Lagrangian, (6 points), (ii) optimality conditions in raw form, (5 points), (iii) derivation of the intratemporal efficiency condition and the Euler equation, (4 points). PLEASE CHECK whether (iii) is DERIVED in one of the later parts, in that case allocate the corresponding points here. XX*

The period-by-period budget constraints of the consumer read

$$\begin{aligned} c_1 + s &= h_1 w_1 \\ c_2 &= h_2 w_2 + (1+r)s. \end{aligned}$$

As a lifetime constraint, the two can be combined to yield

$$c_1 + \frac{c_2}{1+r} = h_1 w_1 + \frac{h_2 w_2}{1+r}.$$

The Lagrangian of the optimization problem reads (sequential formulation)

$$\begin{aligned} \mathcal{L} &= \log(c_1) - \frac{\phi}{2}(h_1)^2 + \beta \left[ \log(c_2) - \frac{\phi}{2}(h_2)^2 \right] \\ &\quad + \lambda_1 [h_1 w_1 - c_1 - s] + \lambda_2 [h_2 w_2 + (1+r)s - c_2], \end{aligned}$$

or (lifetime formulation)

$$\begin{aligned} \mathcal{L} &= \log(c_1) - \frac{\phi}{2}(h_1)^2 + \beta \left[ \log(c_2) - \frac{\phi}{2}(h_2)^2 \right] \\ &\quad + \lambda \left[ h_1 w_1 + \frac{h_2 w_2}{1+r} - c_1 - \frac{c_2}{1+r} \right]. \end{aligned}$$

The optimality conditions are (sequential formulation)

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial c_t} = \beta^{t-1} c_t^{-1} - \lambda_t \\ 0 &= \frac{\partial \mathcal{L}}{\partial h_t} = \beta^{t-1} \phi h_t + \lambda_t \omega_t \\ 0 &= \frac{\partial \mathcal{L}}{\partial s} = -\lambda_1 + \lambda_2(1+r) \\ 0 &= \frac{\partial \mathcal{L}}{\partial \lambda_1} = h_1 \omega_1 - c_1 - s \\ 0 &= \frac{\partial \mathcal{L}}{\partial \lambda_2} = h_2 \omega_2 + (1+r)s - c_2, \end{aligned}$$

or (lifetime formulation)

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial c_t} = \beta^{t-1} c_t^{-1} - \frac{\lambda}{(1+r)^{t-1}} \\ 0 &= \frac{\partial \mathcal{L}}{\partial h_t} = \beta^{t-1} \phi h_t + \frac{\lambda \omega_t}{(1+r)^{t-1}} \\ 0 &= \frac{\partial \mathcal{L}}{\partial \lambda} = h_1 \omega_1 + \frac{h_2 \omega_2}{1+r} - c_1 - \frac{c_2}{1+r}. \end{aligned}$$

In any case, the optimality conditions can be summarized by the intratemporal optimality condition

$$\phi h_t = c_t^{-1} \omega_t,$$

the intertemporal optimality condition (Euler equation)

$$\frac{c_2}{c_1} = \beta(1+r),$$

and the period-by-period constraints

$$\begin{aligned} c_1 &= h_1 \omega_1 - s \\ c_2 &= h_2 \omega_2 + (1+r)s, \end{aligned}$$

or the lifetime budget constraint

$$c_1 + \frac{c_2}{1+r} = h_1 \omega_1 + \frac{h_2 \omega_2}{1+r}.$$

(b) (10 Points) Derive the optimal labor supply and consumption in both periods.

- **Hint 1:** combine first the intratemporal optimality conditions and the Euler equation to derive  $h_2$  and  $c_1$  as a function of  $h_1$  and parameters only. Then use the lifetime budget constraint to derive the optimal labor supply in period 1,  $h_1$ . From here, you can compute the remaining equilibrium variables.

- **Hint 2:** if you were not able to solve part (a), you can assume that optimal consumption and labor supply is characterized through the intratemporal optimality condition

$$\phi h_t = c_t^{-1} w_t,$$

the intertemporal optimality condition (Euler equation)

$$\frac{c_2}{c_1} = \beta(1+r),$$

and the lifetime budget constraint

$$c_1 + \frac{c_2}{1+r} = h_1 w_1 + \frac{h_2 w_2}{1+r}.$$

and you will be able to solve the following parts of this exercise.

**Solution:**

*XX Allocation of points: (i) derivation of optimal consumption levels, or some other intermediate steps (some students directly went after the labor supply in the lifetime budget constraint), (5 points), (ii) derivation of the optimal labor supply, (5 points). PLEASE also allocate points (2 instead of 5) in each step for derivations that are wrong, but go in the right direction and show some understanding of the model. XX*

Combine the intratemporal optimality conditions and the Euler equation to yield the second period labor supply as a function of first period labor supply

$$\frac{h_2}{h_1} = \frac{w_2 c_1}{w_1 c_2} = \frac{w_2}{w_1} [\beta(1+r)]^{-1} \Leftrightarrow h_2 = \frac{w_2}{w_1 \beta(1+r)} h_1.$$

Moreover, consumption in period 1 is then given by

$$c_1 = \frac{w_1}{\phi h_1}.$$

Combining the Euler equation,  $c_2 = \beta(1+r)c_1$ , with the lifetime budget constraint yields the first period labor supply

$$\begin{aligned} c_1 + \frac{\beta(1+r)c_1}{1+r} &= h_1 w_1 + \frac{h_2 w_2}{1+r} \\ \Leftrightarrow c_1(1+\beta) &= h_1 w_1 + \frac{\frac{w_2}{w_1 \beta(1+r)} h_1 w_2}{1+r} \\ \Leftrightarrow \frac{w_1(1+\beta)}{\phi h_1} &= h_1 \left( w_1 + \frac{(w_2)^2}{w_1 \beta(1+r)^2} \right) \\ \Leftrightarrow \frac{1+\beta}{\phi} &= (h_1)^2 \left( 1 + \beta^{-1} \left( \frac{w_2}{w_1(1+r)} \right)^2 \right) \\ \Leftrightarrow h_1 &= \left[ \frac{1+\beta}{\phi} \left( 1 + \beta^{-1} \left( \frac{w_2}{w_1(1+r)} \right)^2 \right)^{-1} \right]^{1/2}. \end{aligned}$$

The optimal second period labor supply is then

$$\begin{aligned} h_2 &= \frac{w_2}{w_1\beta(1+r)}h_1 \\ &= \left[ \frac{1+\beta}{\phi} \beta^{-2} \left( \frac{w_2}{w_1(1+r)} \right)^2 \left( 1 + \beta^{-1} \left( \frac{w_2}{w_1(1+r)} \right)^2 \right)^{-1} \right]^{1/2} \\ &= \left[ \frac{1+\beta}{\phi} \left( \beta^2 \left( \frac{w_2}{w_1(1+r)} \right)^{-2} + \beta \right)^{-1} \right]^{1/2}. \end{aligned}$$

Optimal consumption follows immediately

$$c_1 = \frac{w_1}{\phi h_1} \quad (5)$$

$$c_2 = \beta(1+r)c_1. \quad (6)$$

- (c) (5 Points) Suppose now that the considered economy was closed, so that the interest rate is endogenously determined within the country and assume that there is no capital and no bond supply in the economy (so, **the equilibrium savings must be zero**). The representative firm produces with the production function

$$Y_t = A_t H_t,$$

where  $H_t$  is the firm's labor demand in period  $t$  and input factor markets are competitive. What is the equilibrium wage rate in both periods,  $t \in \{1, 2\}$ ?

**Solution:**

Equilibrium wages are given by the marginal product of labor

$$w_t = \frac{\partial Y_t}{\partial H_t} = A_t,$$

which is a function of labor productivity,  $A_t$ , only (XX allocate full or no points XX).

- (d) (10 Points) Still, consider the closed economy described in part (c) where equilibrium savings must be zero. Furthermore, assume that the consumer anticipates in period  $t = 1$  a boom in the second period  $t = 2$ , this means  $A_2$  increases to  $A'_2 = 2A_2$ . How do the optimal labor supply and consumption in both periods change compared to the scenario where the productivity was still  $A_2$ ? How does the wage in the two periods change? Explain the intuition of your findings (hint: you will be able to solve this exercise even if you struggled before. Work with the intratemporal optimality condition stated in part (b) and the period-by-period constraints

$$\begin{aligned} c_1 &= h_1 w_1 - s \\ c_2 &= h_2 w_2 + (1+r)s, \end{aligned}$$



and anticipate the equilibrium savings behavior,  $s$ .)

**Solution:**

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Relative wages across periods are given by the relative productivity

$$\frac{w_2}{w_1} = \frac{A_2}{A_1},$$

thus if the productivity in the second period doubles, also the wage in the second period will double. The relative wage will be double of what it has been before (XX 2 points. XX). With a zero asset supply, the period-by-period constraints imply

$$c_t = h_t w_t = \frac{w_t c_t^{-1}}{\phi} w_t \Leftrightarrow c_t = \frac{w_t}{\phi^{1/2}},$$

such that the labor supply is given by

$$h_t = \frac{w_t c_t^{-1}}{\phi} = \phi^{1/2}.$$

Thus, the labor supply does not respond to relative price changes across periods (XX derivation and interpretation labor supply, 4 points XX). And as the equilibrium savings have to be zero (the equilibrium interest rate will adjust), the only margin of adjustment for the consumer will be to adjust relative consumption across periods in the same fashion as relative productivity. Thus, second period consumption must fall (XX correct adjustment and interpretation of consumption, 4 points XX).

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- (e) (10 Points) Consider the same economy and experiment as before that is a boom in the second period  $t = 2$ , this means  $A_2$  increases to  $A_2' = 2A_2$ . But now assume that the economy is small open such that the consumer can save at the fixed world interest  $r$  (like in the small open economy considered in parts (a) and (b)). How does the optimal labor supply in both periods and optimal consumption in the first period change? Explain intuitively what changes relative to the closed economy case discussed in part (d). (**hint:** if you did not derive the optimal labor supply in part (b), then you can assume that they are given by

$$h_1 = \left[ \frac{1+\beta}{\phi} \left( 1 + \beta^{-1} \left( \frac{w_2}{w_1(1+r)} \right)^2 \right)^{-1} \right]^{1/2} \quad (7)$$

$$h_2 = \left[ \frac{1+\beta}{\phi} \left( \beta^2 \left( \frac{w_2}{w_1(1+r)} \right)^{-2} + \beta \right)^{-1} \right]^{1/2}, \quad (8)$$

and you will still be able to answer this question.)

**Solution:**

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According to Equations (7) and (8) labor supply in the first (second) period falls (increases) as the relative wage  $w_2/w_1$  increases (XX correct adjustment, 4 points XX).

Moreover, Equation (5) implies that the optimal consumption in the first period will increase (*XX correct adjustment, 2 points XX*).

The main difference to the closed economy case is that the agent can shift resources across time by saving resources (and working more) in the period with a high productivity for the period with a low productivity. In the closed economy however, this not possible since the asset supply is restricted to be zero and the interest rate will instead adjust to bring the economy into an equilibrium (*XX correct comparison, 4 points XX*).

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- (f) (10 Points) Consider again the closed economy setup where the asset supply is zero and the interest rate endogenous. Assume that in the second period the government decides to subsidize labor income at rate  $\tau = 100\%$ , that is for every NOK you earn the government gives you one NOK in addition, so that the after-tax wage for the consumer in the second period is doubled. How does labor supply and consumption in both periods change? How do the wages paid by the firm in the two periods change? Explain the intuition of your findings. (hint: the answer does not involve any additional math.)

**Solution:**

The wages paid by the firm are unchanged, they are still equal to the marginal product of labor,  $w_t = A_t$  (*XX 2 points XX*). However, the lifetime income of the agent is affected in the same way as if her productivity increased to  $2A_2$ . Thus, the results are the same as in part (d). The labor supply does not respond the labor income tax (*XX right adjustment, 4 points XX*), and the consumption is only increased in the second period (*XX right adjustment, 4 points XX*).

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