Problem Set 5: Overlapping Generations and Pensions (Solution)

Exercise 5.1: A simple life-cycle overlapping generations model

Consider a representative consumer who lives for only two periods denoted by $t = 1, 2$. The consumer is born in period 1 without any financial assets and leaves no bequests nor debt at the end of period 2. The consumer's labor income is $w_t \geq 0$ in each period and her preferences over consumption can be represented by the utility function

$$U(c_1, c_2) = u(c_1) + \beta u(c_2), \tag{1}$$

where the momentary utility function is given by

$$u(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta}, & \theta \neq 1. \\ \log(c), & \theta = 1, \end{cases}$$

with $\theta > 0$. For the moment we abstract from the production side of the economy and simply assume that the consumer can borrow and lend consumption across periods at the given real interest rate, $r > 0$.

(a) Write down the consumer's net present value budget constraint, and find the optimal consumption and savings over the life-cycle.
Solution:

The agent's budget constraints in the two periods read (implicitly assuming the terminal condition that savings in the second period must be equal to zero)

$$c_1 + s = w_1$$
$$c_2 = (1+r)s + w_2 \quad \Leftrightarrow \quad s = \frac{c_2 - w_2}{1+r}.$$

Substituting out the savings, s, yields the net present value private budget constraint

$$c_1 + \frac{c_2}{1+r} = w_1 + \frac{w_2}{1+r}. \tag{2}$$

Equation (2) formalizes that consumption can be shifted across periods but is bounded by the consumers discounted lifetime income. Maximizing $U(c_1, c_2)$ subject to the lifetime budget constraint in Equation (2) yields the first-order optimality conditions for consumption (let λ denote the Lagrange multiplier on the lifetime budget constraint)

$$0 = u'(c_1) - \lambda$$
$$0 = \beta u'(c_2) - \lambda/(1+r).$$

Combining the two yields the consumption Euler equation

$$c_2/c_1 = [\beta(1+r)]^{1/\theta}. \tag{3}$$

Combining Equations (2) and (3) yields

$$c_1 + \frac{[\beta(1+r)]^{1/\theta}}{1+r} c_1 = \frac{1 + r + [\beta(1+r)]^{1/\theta}}{1+r} c_1 = w_1 + \frac{w_2}{1+r},$$

which can be reformulated as first period consumption

$$c_1 = \frac{1+r}{(1+r) + [\beta(1+r)]^{1/\theta}} \left(w_1 + \frac{w_2}{1+r} \right) = \frac{1}{1 + \beta^{1/\theta}(1+r)^{1/\theta - 1}} \left(w_1 + \frac{w_2}{1+r} \right),$$

implying that future consumption must be be

$$c_2 = \frac{[\beta(1+r)]^{1/\theta}}{1 + \beta^{1/\theta}(1+r)^{1/\theta - 1}} \left(w_1 + \frac{w_2}{1+r} \right).$$

The optimal savings follow immediately

$$\begin{aligned} s &= w_1 - c_1 \\ &= w_1 \left(\frac{1 + \beta^{1/\theta}(1+r)^{1/\theta-1}}{1 + \beta^{1/\theta}(1+r)^{1/\theta-1}} - \frac{1}{1 + \beta^{1/\theta}(1+r)^{1/\theta-1}} \right) - \frac{1}{1 + \beta^{1/\theta}(1+r)^{1/\theta-1}} \frac{w_2}{1+r} \\ &= \frac{1}{1 + \beta^{1/\theta}(1+r)^{1/\theta-1}} \left(\beta^{1/\theta}(1+r)^{1/\theta-1} w_1 - \frac{w_2}{1+r} \right). \end{aligned}$$

(b) Discuss the effect of an increase in the gross real interest rate $1 + r$ (remember that this corresponds to an increase in the price of c_1 in terms of c_2) on first-period consumption c_1. Discuss the income, substitution, and wealth effect of this price change and how they relate to the EIS.
Solution:

Write first period consumption as

$$c_1 = \frac{1}{1 + \beta^{1/\theta}(1+r)^{1/\theta-1}} \left(w_1 + \frac{w_2}{1+r} \right),$$

so the first factor only contains one term involving the real interest rate. The derivative with respect to $1 + r$ then yields

$$\begin{aligned} \partial c_1 / \partial(1+r) = &- \frac{(1/\theta - 1)\beta^{1/\theta}(1+r)^{1/\theta-2}}{\left[1 + \beta^{1/\theta}(1+r)^{1/\theta-1}\right]^2} \left(w_1 + \frac{w_2}{1+r} \right) \\ &- \frac{1}{1 + \beta^{1/\theta}(1+r)^{1/\theta-1}} \frac{w_2}{(1+r)^2}. \end{aligned}$$

- The first term of this derivative captures the substitution and income effect of this price change. Depending on the value of the EIS $= 1/\theta$, this compound effect on first period consumption can be positive (EIS < 1, the income effect dominates) or negative (EIS > 1, if the substitution effect dominates).

- The second term captures the wealth effect which is always weakly negative as an increase in the real interest rate weakly decreases the agent's present value lifetime income. Note that this effect only shows up in the intertemporal context, and not on the static micro context where the value of income is simply given and does not change with the relative price.
- For high enough values of the EIS first period consumption will fall and savings increase in response to an increase in $1 + r$. Consider for example $\theta \to 1$ (EIS =1), then the first term disappears (income and substitution effect exactly offset each other with logarithmic preferences!) and the price effect will be unambiguously negative as long as $w_2 > 0$.
- $\theta \to \infty$ (EIS =0) corresponds to the case where the substitution effect is zero (as the agent wants to consume in fixed proportions, see the Euler equation), and the income effect dominates completely the first term.