1 Lecture Notes: The Ramsey Model

1.1 Deriving the Key Equations in the Ramsey Model

Purpose of lecture: amend the Solow model with endogenous choices generated by preferences of individuals, i.e.,

- Individual Preferences

$$\sum_{t=1}^{T} \beta^t u(c_t)$$

  and each period they face a budget constraint

$$\begin{aligned} c_t + a_{t+1} &= (1 + r_t) a_t + w_t \\ a_{T+1} &\geq 0, \end{aligned}$$

  where $\beta < 1$. The constraint $a_{T+1} \geq 0$ rules out Ponzi/Meadoff schemes. Given complete markets and no Ponzi (Meadoff) schemes, the borrowing constraints can be written as one constraint, where the NPV of consumption is equal to the present value of wages plus current financial wealth:

$$\sum_{t=1}^{T} \frac{c_t}{\Pi_{j=1}^{t}(1 + r_j)} \leq a_1 + \sum_{t=1}^{T} \frac{w_t}{\Pi_{j=1}^{t}(1 + r_j)} \tag{1}$$

- Perfect foresight:
  - Note: future wages and interest rates matter for the optimal decision
  - Key question: what does the individuals expect about the future?
  - Answer: they have rational expectations about future prices. Otherwise, a rational individual can achieve systematic arbitrage and drive the irrational individuals out of the market (arrange bets on future realizations; Friedman, 1953)

- Under no uncertainty, *rational expectations* imply *perfect foresight* about the future. Thus, agents solve

$$\max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{T} \beta^t u(c_t),$$

  subject to (1)

  - Set the problem up as a Lagrangian problem

$$\max_{\{c_t\}_{t=1}^{T}} \Lambda = \sum_{t=1}^{T} \beta^t u(c_t) + \lambda \left(a_1 + \sum_{t=1}^{T} \frac{w_t}{\Pi_{j=1}^{t}(1 + r_j)} - \sum_{t=1}^{T} \frac{c_t}{\Pi_{j=1}^{t}(1 + r_j)} \right)$$

First-order conditions are

$$\beta^t u'(c_t) - \lambda \frac{1}{\Pi_{j=1}^t (1 + r_j)} = 0$$

Combine (by taking the ration on both sides) the FOC for period t and period $t+1$ to archieve the *Euler equation*:

$$\begin{aligned}
\frac{\beta^t u'(c_t)}{\beta^{t+1} u'(c_{t+1})} &= \frac{\lambda \frac{1}{\Pi_{j=1}^t (1+r_j)}}{\lambda \frac{1}{\Pi_{j=1}^{t+1} (1+r_j)}} \\
&\Rightarrow \\
\frac{u'(c_t)}{\beta u'(c_{t+1})} &= \frac{(1 + r_{t+1}) \cdot \Pi_{j=1}^t (1 + r_j)}{\Pi_{j=1}^t (1 + r_j)} \\
&\Rightarrow \\
\frac{u'(c_t)}{u'(c_{t+1})} &= \beta (1 + r_{t+1}),
\end{aligned}$$

and then solve for r_{t+1} using the optimality condition for firms, $r_{t+1} = f'(k_{t+1}) - \delta$.

- Alternatively: formulate problem as planner's problem (it is natural to assume that the planner has perfect foresight)

$$\begin{aligned}
& \max_{\{c_t\}_{t=1}^T} \sum_{t=1}^T \beta^t u(c_t), \\
& \text{subject to} \\
c_t + k_{t+1} &= f(k_t) + (1 - \delta) k_t \\
k_{T+1} &\geq 0
\end{aligned}$$

  - Write it as a Lagrangian:

$$\max_{\{c_t\}_{t=1}^T} \Lambda = \sum_{t=1}^T \beta^t \{u(c_t) + \lambda [f(k_t) + (1 - \delta) k_t - c_t - k_{t+1}]\}$$

First-order conditions are

$$\begin{aligned}
\frac{\partial \Lambda}{\partial c_t} &= \beta^t \{u'(c_t) - \lambda_t\} = 0 \\
\frac{\partial \Lambda}{\partial k_{t+1}} &= -\lambda_t \beta^t + \beta^{t+1} \lambda_{t+1} \{f'(k_{t+1}) + (1 - \delta)\} = 0
\end{aligned}$$

Combining the FOCs yield

$$\beta \{f'(k_{t+1}) + 1 - \delta\} = \frac{\lambda_t}{\lambda_{t+1}} = \frac{u'(c_t)}{u'(c_{t+1})}$$

With $u(c) = c^{1-\gamma}/(1-\gamma)$, the Euler equation becomes

$$\begin{aligned}
\frac{(c_t)^{-\gamma}}{(c_{t+1})^{-\gamma}} &= \beta \left\{ f'(k_{t+1}) + 1 - \delta \right\} \\
&\Rightarrow \\
\frac{c_{t+1}}{c_t} &= \left[\beta \left\{ f'(k_{t+1}) + 1 - \delta \right\} \right]^{\frac{1}{\gamma}}
\end{aligned}$$

1.2 Characterize the solution

- Solution is characterized by three equations

  1. The Euler equation

  $$\frac{c_{t+1}}{c_t} = \left(\beta \left(f'(k_{t+1}) + 1 - \delta \right) \right)^{\frac{1}{\gamma}} \tag{2}$$

  2. Resource constraint

  $$k_{t+1} = f(k_t) + (1-\delta) k_t - c_t \tag{3}$$

  3. The terminal condition $k_{T+1} \geq 0$. Note that when $T \to \infty$, there is no "last period" and the terminal condition gets replaced by a "tansversality condition":

  $$\lim_{T \to \infty} \beta^t u'(c_t) = 0$$

  – Consider $T = 2$. There are 2 variables each period, (c_t, k_t), so four unknowns. The initial k_1 is predetermined, the two equations give relationships between (c_t, k_t) and (c_{t+1}, k_{t+1}). Finally the transversality condition provides the last equation).

- **Uniqueness**: The social-planner problem has a unique solution. Due to the Welfare Theorems, the competitive equilibrium must therefore also be unique. This suggests that the competitive equilibrium is also unique in the infinite horizon case when $T \to \infty$. (this can be proven formally when $\beta < 1$)

- Analyze the solution in three ways:

  1. Steady-state analysis (today), focusing on the infinite horizon case when $T \to \infty$
  2. Dynamics:
     (a) Phase diagram (today)

1.3 Steady State Analysis

- Assume there exists a steady state (where all variable grow at the same rate). Since there is no technical growth, it is natural to guess zero growth in the long run (otherwise, capital accumulation would drive r_t to zero)
- Set $c_{t+1} = c_t = c^*$ and $k_{t+1} = k_t = k^*$ in equations (2)-(3):

$$\frac{c^*}{c^*} = \left(\beta \left(f'\left(k^*\right) + 1 - \delta\right)\right)^{\frac{1}{\gamma}}$$

$$k^* = f\left(k^*\right) + \left(1 - \delta\right) k^* - c^*$$

which implies

$$k^* = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta}\right)^{\frac{1}{1-\alpha}}$$

- Note that k^* increases in β and α and decreases in δ (why?)
- Note that this is smaller than the golden rule:

$$k^* \leq k^g = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}$$

  - Why?? Answer: becuase with discounting agents require a compensation to wait (i.e., $r > 0$). Note that when $\beta = 1$, there is no difference: $k^* = k^g$

- Stability: need to prove that the steady state is stable (note: we ignored the trivial and uninteresting steady state with $c = k = 0$). By combining equations (2)-(3) we get a second-order difference equation in k:

$$c_t = f\left(k_t\right) + \left(1 - \delta\right) k_t - k_{t+1}$$

$$f\left(k_{t+1}\right) + \left(1 - \delta\right) k_{t+1} - k_{t+2} = \left[\beta \left(f'\left(k_{t+1}\right) + 1 - \delta\right)\right]^{\frac{1}{\gamma}} \cdot \left[f\left(k_t\right) + \left(1 - \delta\right) k_t - k_{t+1}\right] \quad (4)$$

It turns out that equation (4) is stable iff $\beta < 1$ ()

1.4 Dynamics

- Fundamental problem: given k_1 (a state variable), what is the optimal choice of c_1? Note that the Euler equation determines the whole sequence of $\{c_t\}_{t=1}^{\infty}$ if we know the initial c_1. Simple tool: Phase diagram of the (k_t, c_t) space:
  - Consider the choices of c_1 (given initial k_1) such that capital is constant (i.e., $k_2 = k_1$) in the resource constraint:

$$\begin{aligned} k_2 &= k_1 = f\left(k_1\right) + \left(1 - \delta\right) k_1 - c_1 \\ &\Rightarrow \\ c_1 &= f\left(k_1\right) - \delta k_1 \end{aligned}$$

... hump-shaped graph originating in origin.

– Consider the choices of c_1 (given initial k_1) such that consumption is constant (i.e., $c_2 = c_1$) in the Euler equation:

$$\begin{aligned} \frac{c_2}{c_1} &= \frac{c_1}{c_1} = \left(\beta\left(f'\left(k_2\right)+1-\delta\right)\right)^{\frac{1}{\gamma}} \\ &\Rightarrow \\ 1 &= \beta\left(f'\left(k_2\right)+1-\delta\right) \\ \frac{1}{\beta}-1+\delta &= f'\left(k_2\right)=\alpha\left(k_2\right)^{\alpha-1} \\ k_2 &= k^* \end{aligned}$$

so the combinations of (c_1, k_1) such that consumption is constant (according to the Euler equation) are given by

$$\begin{aligned} k^* &= f\left(k_1\right)+(1-\delta)\,k_1-c_1 \\ &\Rightarrow \\ c_1 &= f\left(k_1\right)+(1-\delta)\,k_1-k^* \end{aligned}$$

– Analysis of dynamics.