Final Exam
ECON 4310, Fall 2017

1. Do **not write with pencil**, please use a ball-pen instead.
2. Please answer in **English**. Solutions without traceable outlines, as well as those with unreadable outlines **do not earn points**.
3. Please start a **new page** for **every** short question and for every subquestion of the long questions.

Good Luck!

	Points	**Max**
Exercise A		**40**
Exercise B		**40**
Exercise C		**60**
Σ		**140**

Grade: __________

Exercise A:
Short Questions (40 Points)

Answer each of the following short questions on a seperate answer sheet by stating as a first answer True/False and then give a short but instructive explanation. You can write, calculate, or draw to explain your answer. You only get points if you have stated the correct True/False *and* provided a correct explanation to the question. We will not assign negative points for incorrect answers.

Exercise A.1: (10 Points) The Optimal Capital Stock

Because more capital allows more output to be produced, it is always better for a country to have more capital stock. True or false?

Your Answer:

True ☐ False: ☒

A per capita capital stock above the golden rule level is so costly to maintain due to depreciation and population growth that reducing the capital stock would ac-tually make it possible to increase consumption in all future periods. The golden rule level is associated with the steady state that maximizes steady-state consumption. Remember that agents in the economy want to maximize their (discounted) utility which depends on consumption, not on maximizing the capital stock.

Exercise A.2: (10 Points) Permanent Income Hypothesis

Consider an individual agent. If her income varies randomly from one period to another, then her consumption will also vary from one period to another, but less so than her income. True or false?

Your Answer:

True: ☐ False: ☒

From the permanent income hypothesis, we know that the response of contemporaneous consumption to changes in income will depend on how much of the change in income is permanent, and how much is transitory. If the movements in income represent permanent movements in expected permanent income, then it is possible that consumption could move one to one with the changes in income.

Exercise A.3: (10 Points) Permanent shocks in Real business cycle model

Consider a simple two-period model of labor supply, as we saw in lectures, where we assume that utility is separable in consumption and labor supply:

$$\max_{\{c_0,c_1,h_0,h_1,a_1\}} \log c_0 - \phi \frac{h_0^{1+\theta}}{1+\theta} + \beta[\log c_1 - \phi \frac{h_1^{1+\theta}}{1+\theta}]$$
$$s.t.$$
$$c_0 + a_1 = w_0 h_0 + (1+r_0)a_0$$
$$c_1 = w_1 h_1 + (1+r_1)a_1$$

for given $a_0 = 0$. We know the household has the following intertemporal labor supply condition:

$$\beta \frac{\phi h_1^\theta}{\phi h_0^\theta} = \frac{w_1}{(1+r_1)w_0},$$

and the solution for h_0 is given by:

$$\phi h_0^{1+\theta} \left[1 + \left(\frac{w_1}{(1+r_1)w_0} \right)^{1+\frac{1}{\theta}} \beta^{-\frac{1}{\theta}} \right] = (1+\beta).$$

Suppose there is a permanent change to wages at the beginning of time 0: both the wages in the first and second period increase by 10%. Then this household will take advantage of this opportunity, and increase his labor supply h_1. True or false?

Your Answer:

True ☐ False: ☒

By inspecting the solution for h_0 we know h_0 will not change; Then by using the intertemporal labor supply condition, we know h_1 and h_0 is proportional to each other. Therefore, labor supply h_1 does not change.

Exercise A.4: (10 Points) Capital and skilled labor complementarity in Solow model

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Consider the following version of the Solow model with two types of workers, skilled workers and unskilled workers. The number of workers are given by S and U, respectively. There are no population growth or technology growth. The production function is specified in (2), and the economy is described by the following equations

$$K_{t+1} = sY_t + (1-\delta)K_t \tag{1}$$

$$Y_t = \left(\phi_1 \left[\phi_2 K_t^{\frac{\rho-1}{\rho}} + (1-\phi_2)S^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}\frac{\sigma-1}{\sigma}} + (1-\phi_1)U^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \tag{2}$$

where $0 < \delta < 1$ is the depreciation rate of physical capital, and the saving rate s is constant. ϕ_1, ϕ_2 ρ, σ are also constant parameters. Denote the share of skilled workers as $\theta \equiv \frac{S}{L} = \frac{S}{S+U}$.

The steady-state level of the capital stock per capita, $k_t \equiv K_t/L$, is described by

$$\delta k^\star = s\left[(k^\star)^\rho + (A_1\theta)^\rho + (A_2(1-\theta))^\rho\right]^{1/\rho}.$$

$$\delta k^\star = \left(\phi_1 \left[\phi_2 (k^\star)^{\frac{\rho-1}{\rho}} + (1-\phi_2)\theta^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}\frac{\sigma-1}{\sigma}} + (1-\phi_1)(1-\theta)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}.$$

True or false?

Your Answer:

True: ☐ False: ☒

In the steady state, we should have $\delta k^\star = sY^\star/L$. Plugging into the production function, note that it is homothetic in the inputs. We can then show

$$\delta k^\star = s\left(\phi_1 \left[\phi_2 (k^\star)^{\frac{\rho-1}{\rho}} + (1-\phi_2)\theta^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}\frac{\sigma-1}{\sigma}} + (1-\phi_1)(1-\theta)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}.$$

and the statement in the problem is not correct. (the saving rate s is missing there.)

Exercise B:
Long Question (40 Points)
OLG Model

Consider an overlapping generations economy that goes on to infinity, but in which individuals live only for two periods each. There are many individuals in the economy and we denote the mass of a cohort of workers by L_t. The size of each cohort remains constant over time, and we can normalize $L_t = 1$. There is no production in this economy. Individuals receive exogenous endowments in each period of life. In particular, let y_t denote the endowment of an individual born at time t. The second period endowment of the same individual (born at time t) $y_t(1+\gamma)$, where γ can be negative. Individuals can save through a storage technology, which pays a fixed return $(1+r)$ next period for every unit of the endowment saved this period. An individual born at time t has preferences over consumption in the two periods of his life:

$$U_t = \log(c_{1t}) + \beta \log(c_{2t+1}))$$

where c_1t is the consumption of an individual born at time t in the first period of his life, and c_{2t+1} is the consumption of the same individual in the second period of his life. $\beta \in (0,1)$ is the subjective discount factor. Finally, assume that first period endowments grow at rate ϕ between successive generations of young, so that $y_t = (1+\phi)t_{t-1}$.

(a) (10 Points) Write down the resource constraint for this economy at time t.

Solution:

The resource constrain tells you how the economy's total resources in a given period is allocated across different uses. In this case the total resources are the sum of the endowments of the young and the old, and the total uses of consumption of the young and the old:

$$\begin{aligned} c_{1t}L_t + c_{2t+1}L_{t+1} =& y_tL_t + y_{t-1}(1+\gamma)L_{t+1} \\ c_{1t} + c_{2t+1} =& y_{t-1}(1+\phi) + y_{t-1}(1+\gamma) \\ c_{1t} + c_{2t+1} =& y_{t-1}(2+\phi+\gamma) \end{aligned}$$

(b) (20 Points) Write down the budget constraints for an individual born at time t. Assuming that the individual maximizes utility, solve for his choice of first period consumption and savings rate, $s_{rt} = s_t/y_t$.

Solution:

The first period budget constraint of an individual born at time t when young is:

$$c_{1t} + s_t = y_t$$

The budget constraint for the same individual when old, in the second period of his life is:

$$c_{2t+1} = (1+r)s_t + y_t(1+\gamma)$$

Solve the first period budget constrain for s_t and substitute into the 2nd one to get the lifetime budget constraint:

$$c_{1t} + \frac{c_{2t+1}}{1+r} = y_t[1 + \frac{1+\gamma}{1+r}]$$

The Langrangian for the individual's problem is:

$$\mathcal{L} = \log(c_{1t}) + \beta \log(c_{2t+1}) + \lambda\Big(y_t[1 + \frac{1+\gamma}{1+r}] - c_{1t} - \frac{c_{2t+1}}{1+r}\Big)$$

Taking the first order conditions, you can derive the Euler equation:

$$c_{2t+1} = \beta(1+r)c_{1t}$$

Substitute the Euler into the lifetime budget constraint to solve for the individual's choice of first period consumption.

$$c_{1t} = y_t\Big[\frac{2+r+\gamma}{(1+r)(1+\beta)}\Big]$$

The individual's savings rate is then the fraction of first period income (endowment) saved:

$$s_{rt} = \frac{s_t}{y_t} = \frac{y_t - c_{1t}}{y_t} = 1 - \frac{c_{1t}}{y_t} = 1 - \Big[\frac{2+r+\gamma}{(1+r)(1+\beta)}\Big]$$

(c) (10 Points) How does the growth of income over the life-cycle, γ, expected by one individual affect his savings rate? Explain. How does an increase in ϕ affect the growth of aggregate savings in this economy? Explain.

Solution:

To see the effect of life-time income growth (or decline) on an individual born in time t, calculate the derivative of s_{rt} with respect to γ:

$$\frac{\partial s_{rt}}{\partial \gamma} = \Big[\frac{-1}{(1+r)(1+\beta)}\Big]$$

Intuitively when income grows more over the life-cycle there is less need for saving when young. On the other hand the aggregate savings rate in the economy is constant and not affected by ϕ. Aggregate savings at time $t+1$ is $S_{t+1} = s_r y_{t+1} L$. The growth rate of aggregate savings is:

$$\frac{S_{t+1}}{S_t} = \frac{s_r y_{t+1} L}{s_r y_t L} = \frac{y_{t+1}}{y_t} = 1 + \phi$$

The savings rate is, however, independent of ϕ.

Exercise B:
Long Question (60 Points)
Temporary fiscal shocks in the Ramsey model

Consider a discrete-time version of Ramsey's growth model. The economy is closed and we consider a representative agent with the following preferences over consumption

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{3}$$

where c_t denotes period t consumption and $\beta \in (0, 1)$ is the subjective discount factor. The momentary utility function is of the form

$$u(c_t) = \frac{c_t^{1-\theta} - 1}{1-\theta},$$

with $\theta > 1$. Every period the agent earns a wage w_t (the labor supply is exogenously set to 1 unit), an interest $r_t a_t$ from her assets holdings and she is subject to the lump-sum tax τ_t. In equilibrium, the agent will choose the sequence consumption and asset holdings $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ to maximize U subject to the period-by-period budget constraint

$$c_t + a_{t+1} = w_t + (1 + r_t)a_t - \tau_t, \tag{4}$$

for a given a_0. The agent is atomic and her decisions do not influence aggregate variables, thus she takes the sequence of taxes, wage rates and interest rates as given.

Firms:

The firm is atomic and acts as a price-taking profit maximizer. Capital can be rented at the rental rate $R_t = r_t + \delta$ (note that the depreciation rate δ is the difference between the rental rate and the interest rate) while labor costs w_t.

The representative firm demands physical capital k_t and labor n_t to produce output y_t with the Cobb-Douglas technology

$$y_t = k_t^{\alpha} n_t^{1-\alpha}. \tag{5}$$

Governments:

The government can raise lump-sum taxes τ_t and rolls over debt in the form of one-period bonds, D_{t+1}, to finance government expenditure, G_t. As it pays an interest rate r_t on the outstanding debt, D_t, the government faces a period-by-period budget constraint

$$G_t = \tau_t + D_{t+1} - (1 + r_t)D_t. \tag{6}$$

Moreover, assume that the time path of government debt is such that it is growing at a lower rate than the interest rate

$$\lim_{T\to\infty} \frac{D_{T+1}}{\Pi_{s=0}^{T}(1 + r_s)} = 0.$$

In other words, it is not feasible for the government to finance the outstanding debt (plus interest payments) by issuing ever more debt as time goes by.

(a) Formulate the Lagrangian of the agent's decision problem (it is common to use λ_t as the Lagrange multiplier on the period t budget constraint).

Derive the first-order conditions for the optimal choice of c_t and a_{t+1}. (15 Points)

Solution:

The Lagrangian of the constrained optimization problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[(w_t + (1+r_t)a_t - \tau_t - c_t - a_{t+1} \right]$$

yielding the following first-order conditions for optimal c_t and a_{t+1}

$$0 = \partial\mathcal{L}/\partial c_t = \beta^t u'(c_t) - \lambda_t$$
$$0 = \partial\mathcal{L}/\partial a_{t+1} = -\lambda_t + (1+r_{t+1})\lambda_{t+1}.$$

(b) Derive the Consumption Euler Equation. (5 Points)

Solution:

Eliminate the Lagrange multiplier λ_t by combining the two and derive the Euler equation for consumption

$$\beta^t u'(c_t) = \beta^{t+1} u'(c_{t+1})(1+r_{t+1}) \quad \Leftrightarrow \quad \frac{\beta u'(c_{t+1})}{u'(c_t)} = \frac{1}{1+r_{t+1}}. \tag{7}$$

This is just standard micro theory. The marginal rate of substitution between tomorrow's and today's consumption (the left-hand side) has to be equal to the relative price of tomorrow's in terms of today's consumption (the right-hand side). Relative price interpretation: the agent needs to save $1/(1+r_{t+1})$ units of today's consumption in assets a_{t+1} to yield 1 unit of consumption tomorrow).

(c) Find the first-order conditions for the firm's optimization problem. Show that firms earn zero profits. (10 Points)

Solution:

The firm's optimization problem is static, and it simply maximizes period-by-period profits

$$\pi_t = k_t^{\alpha} n_t^{1-\alpha} - R_t k_t - w_t n_t.$$

First-order conditions with respect to k_t and n_t are

$$0 = \partial\pi_t/\partial k_t = \alpha(k_t/n_t)^{\alpha-1} - R_t$$
$$0 = \partial\pi_t/\partial n_t = (1-\alpha)(k_t/n_t)^{\alpha} - w_t,$$

implying that input factors are paid their marginal product in equilibrium.

(d) Use the government's budget constraint in Equation (6) and substitute for D_t iteratively ($t = 1, 2, 3, ...$) to derive the government's intertemporal budget constraint in net present value (NPV) terms

$$D_0 = \sum_{t=0}^{\infty} \frac{\tau_t - G_t}{\Pi_{s=0}^{t}(1 + r_s)}. \tag{8}$$

Give an interpretation of Equation (8).

(Hint: you could start out with $D_0 = \frac{1}{1+r_0}[\tau_0 - G_0 + D_1]$, and find a similar expression for D_1 and substitute it into the expression for D_0; do it iteratively.) (10 Points)

Solution:

Just follow the instructions in the problem. Start out with

$$D_0 = \frac{1}{1 + r_0}[\tau_0 - G_0 + D_1].$$

Then insert for D_1 using the same formula

$$\begin{aligned} D_0 &= \frac{1}{1 + r_0}\left[\tau_0 - G_0 + \frac{1}{1 + r_1}[\tau_1 - G_1 + D_2]\right] \\ &= \frac{\tau_0 - G_0}{1 + r_0} + \frac{\tau_1 - G_1}{(1 + r_0)(1 + r_1)} + \frac{D_2}{(1 + r_0)(1 + r_1)} \\ &= \sum_{t=0}^{1} \frac{\tau_t - G_t}{\Pi_{s=0}^{t}(1 + r_s)} + \frac{D_{1+1}}{\Pi_{s=0}^{1}(1 + r_s)}, \end{aligned}$$

and continue until period T to get

$$D_0 = \sum_{t=0}^{T} \frac{\tau_t - G_t}{\Pi_{s=0}^{t}(1 + r_s)} + \frac{D_{T+1}}{\Pi_{s=0}^{T}(1 + r_s)}.$$

Finally, let $T \to \infty$ to yield Equation (8). Thus, the NPV of government expenditures cannot exceed the NPV of lump-sum taxes net of the initial debt position.

Dynamics:

In this economy we know that the solution to the social planner's problem is equivalent to the competitive market equilibrium. According to the social planner's solution, the same consumption Euler equation and resource constraint (goods market clearing) along with the so-called transversality condition (which stands in for the no-Ponzi condition)

$$\begin{aligned} \frac{c_{t+1}}{c_t} &= [\beta(1 + r_{t+1})]^{1/\theta} = \left[\beta(1 + \alpha k_{t+1}^{\alpha-1} - \delta)\right]^{1/\theta} \\ k_{t+1} - k_t &= k_t^{\alpha} - \delta k_t - c_t - G_t \\ \lim_{t\to\infty} \beta^t c_t^{-\theta} k_{t+1} &= 0 \end{aligned}$$

determine the optimal solution of the dynamic system. Let us assume that $G_t = G$, then we can define two correspondances. One which characterizes all possible combinations of (c_t, k_t) when consumption is constant,

$$\mathcal{C}_1(k) \equiv \left\{ c \in [0, \infty) : c_{t+1}/c_t = \left[\beta(1 + \alpha k^{\alpha-1} - \delta)\right]^{1/\theta}, c_{t+1} = c_t = c \right\},$$

and one which captures all combinations if the physical capital stock is constant,

$$\mathcal{C}_2(k) \equiv \{c \in [0, \infty) : c = k_t^a - (k_{t+1} - (1-\delta)k_t) - G, k_{t+1} = k_t = k\}.$$

(e) Draw the two correspondances, $\mathcal{C}_1(k)$ and $\mathcal{C}_2(k)$, in a diagram with k on the horizontal axis and c on the vertical axis, the so called phase diagram. (10 Points)

Solution:

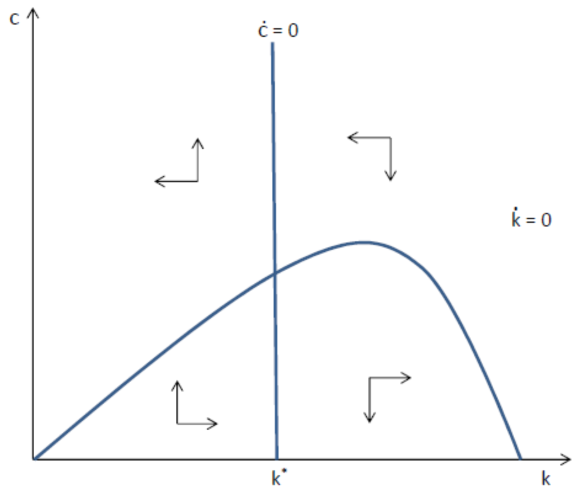

Figure 1: Correspondances for $\mathcal{C}_1(k)$ and $\mathcal{C}_2(k)$

(f) Now assume that the economy has run for a long time and is in its steady state with constant government expenditures, $G_t = G$, and tax policy, $\tau_t = \tau$.

Consider an unexpected and temporary increase of ΔG in government expenditures from period t_0 until period t_1. Sketch the dynamics of consumption and physical capital back to the steady-state in the phase diagram. (10 Points)

Solution:

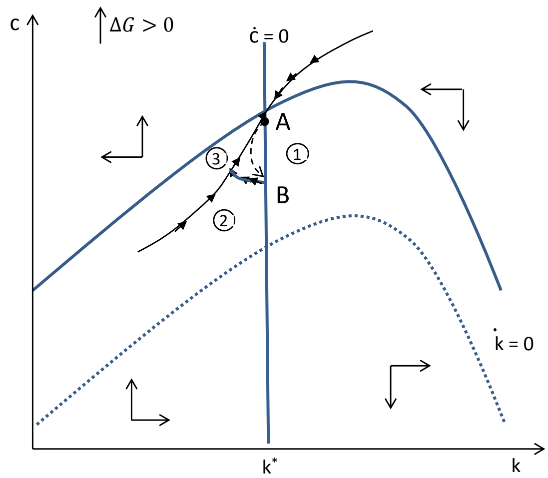

FIGURE 3.1

Figure 2: Dynamics of consumption and physical capital