Compulsory Assignment: ECON 4310

Consumption over the Life Cycle

A finite horizon problem with perfect foresight

Instruction

Please deliver the assignment in Fronter as a typed document in PDF format (no handwriting). Please also attach your files used for computation (e.g. code, and/or excel spreadsheets).

Part I: theory

Consider the following problem of intertemporal consumption choice in discrete and finite time, $t = 0, 1, \ldots, T < \infty$,

$$\max \sum_{t=0}^{T} \beta^t u(c_t), \quad \text{s.t.} \quad a_{t+1} = (1 + r_t)a_t + w_t - c_t, \, \forall t,$$

with $u(c) = \frac{c^{1-\theta}-1}{1-\theta}, \theta > 1$, and $a_0 > 0$, $\{r_t\}_{t=0}^{T}$, $\{w_t\}_{t=0}^{T}$ given, and $0 < \beta < 1$. The trivial solution to this problem is to set the terminal debt to infinity, $a_{T+1} = -\infty$, and enjoying infinite utility. To rule out such a solution it is necessary to impose the additional constraint

$$a_{T+1} \geq 0, \tag{1}$$

which means that the agent cannot leave debt at the end of her life. Note that the terminal condition stated in Equation (1) is the analog of the no-Ponzi condition in the corresponding infinite horizon problem.

(a) State the Lagrangian of this optimization problem. (Hint: use λ_t as the Lagrange multiplier on the period-by-period budget constraint and treat the constraint in Equation (1) as a regular equality constraint with Lagrange multiplier μ). [10 points]

(b) State the first-order optimality conditions with respect to c_t, a_{t+1} (for $t < T$), and a_{T+1}.

Remember to use so called complementary slackness condition (from the Karush-Kuhn-Tucker conditions)

$$\mu a_{T+1} = 0, \, \mu \geq 0.$$

This condition simply means that either $\mu = 0$ and $a_{T+1} \geq 0$ (the inequality constraint is not binding) or $\mu > 0$ and $a_{T+1} = 0$ (the inequality constraint is binding). [10 points]

(c) Given that marginal utility is positive, $u'(c) > 0$, show that $\mu > 0$ implying that $a_{T+1} = 0$. Next, show that as $T \to \infty$

$$\lim_{T\to\infty} \beta^T u'(c_T) a_{T+1} = 0.$$

Comment on this equilibrium condition. [5 points]

(d) Derive the *Consumption* Euler equation of this dynamic system. [5 points]

Part II: numerical exercise

In Part II you will have to do the calculations with your preferred software (Excel, Matlab, R, or any other software).

(e) Set the parameters of the model to $\beta = 97/100$, $T = 44$, $\theta = 3$, $a_0 = 0$, $r_t = 4/100$, $w_t = 1/2$, and use the Euler equation as well as the period-by-period budget constraint to solve for the optimal consumption and asset accumulation paths.

(Hint: guess a c_0 and then solve the dynamic system forward. If $a_{T+1} > 0$, then you need to increase the initial guess c_0 as leaving assets on the table at the end of your life instead of consuming them cannot be optimal. On the other hand, if $a_{T+1} < 0$, then you need to lower c_0. **You may want to use a solver which is available in most spreadsheet applications to solve for the optimal value c_0 such that $a_{T+1} = 0$ in one step.**[1]).

Using again your preferred software, plot the optimal consumption and asset accumulation path in a **single** diagram with t on the horizontal axis, and label the two paths in the graph clearly. Report also the following quantities (up to the third decimal digit): $c_{t=0}, c_{t=T}$ and $a_{t=1}, a_{t=T}$. [10 points for the graph and 5 points for the reported quantities]

(f) Suppose that $r_t = 2.25/100$ instead of $r_t = 4/100$; all other parameters are the same as in (e). How do the optimal consumption and asset accumulation paths change? Please solve the problem again and plot the new graph for optimal consumption and asset accumulation paths; explicitly point out and comment on the changes. Report also the following quantities (up to the third decimal digit): $c_{t=0}, c_{t=T}$ and $a_{t=1}, a_{t=T}$. [10 points]

[1] For example some info about the Solver in Excel are available at this link `http://www.excel-easy.com/data-analysis/solver.html`.

(g) Suppose that $w_{t=20} = 0$ and $w_{t=21} = 0$ (imagine the worker is unemployed for two periods); all other parameters are the same as in (e). How do the optimal consumption and asset accumulation paths change? Please solve the problem again and plot the new graph for optimal consumption and asset accumulation paths. Report also the following quantities (up to the third decimal digit): $c_{t=0}, c_{t=T}$ and $a_{t=1}, a_{t=T}$. Explicitly point out and comment on the changes. [10 points]

(h) Suppose now that $\theta = 100$. Compare optimal consumption and asset accumulation paths when $r_t = 2.25/100$ and when $r_t = 4/100$. All other parameters are the same as in (e). Please solve the problem again and plot the new graphs for optimal consumption and asset accumulation paths for these two cases; explicitly point out and comment on the differences from point (f). Report also the following quantities for each case (up to the third decimal digit): $c_{t=0}, c_{t=T}$ and $a_{t=1}, a_{t=T}$. [10 points]