Compulsory Assignment: ECON 4310

Consumption over the Life Cycle; suggested solution

A finite horizon problem with perfect foresight

Part I: theory

Consider the following problem of intertemporal consumption choice in discrete and finite time, $t = 0, 1, \ldots, T < \infty$,

$$\max \sum_{t=0}^{T} \beta^t u(c_t), \quad \text{s.t.} \quad a_{t+1} = (1 + r_t)a_t + w_t - c_t, \forall t,$$

with $u(c) = \frac{c^{1-\theta}-1}{1-\theta}, \theta > 1$, and $a_0 > 0$, $\{r_t\}_{t=0}^{T}$, $\{w_t\}_{t=0}^{T}$ given, and $0 < \beta < 1$. The trivial solution to this problem is to set the terminal debt to infinity, $a_{T+1} = -\infty$, and enjoying infinite utility. To rule out such a solution it is necessary to impose the additional constraint

$$a_{T+1} \geq 0, \tag{1}$$

which means that the agent cannot leave debt at the end of her life. Note that the terminal condition stated in Equation (1) is the analog of the no-Ponzi condition in the corresponding infinite horizon problem.

(a) State the Lagrangian of this optimization problem. (Hint: use λ_t as the Lagrange multiplier on the period-by-period budget constraint and treat the constraint in Equation (1) as a regular equality constraint with Lagrange multiplier μ). [10 points]

Solution:

The Lagrangian reads

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \lambda_t \left[(1 + r_t)a_t + w_t - c_t - a_{t+1}\right] + \mu a_{T+1}.$$

(b) State the first-order optimality conditions with respect to c_t, a_{t+1} (for $t < T$), and a_{T+1}.

Remember to use so called complementary slackness condition (from the Karush-Kuhn-Tucker conditions)

$$\mu a_{T+1} = 0, \ \mu \geq 0.$$

This condition simply means that either $\mu = 0$ and $a_{T+1} \geq 0$ (the inequality constraint is not binding) or $\mu > 0$ and $a_{T+1} = 0$ (the inequality constraint is binding). [10 points]

Solution:

The optimality conditions read (including complementary slackness condition)

$$0 = \frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) - \lambda_t$$
$$0 = \frac{\partial \mathcal{L}}{\partial a_{t+1}} = -\lambda_t + \lambda_{t+1}(1 + r_{t+1}),\ t < T$$
$$0 = \frac{\partial \mathcal{L}}{\partial a_{T+1}} = -\lambda_T + \mu, \quad \mu a_{T+1} = 0, \quad \mu \geq 0, a_{T+1} \geq 0.$$

(c) Given that marginal utility is positive, $u'(c) > 0$, show that $\mu > 0$ implying that $a_{T+1} = 0$. Next, show that as $T \to \infty$

$$\lim_{T\to\infty} \beta^T u'(c_T) a_{T+1} = 0.$$

Comment on this equilibrium condition. [5 points]

Solution:

Firstly, the optimality conditions imply that $\mu = \beta^T u'(c_T) > 0$, as β^T remains strictly positive for $T < \infty$. Secondly, it follows from the complementary slackness condition that $a_{T+1} = 0$. This is true for any T, and when we have the limiting case it is also true:

$$\lim_{T\to\infty} \beta^T u'(c_T) a_{T+1} = 0,$$

as $T \to \infty$. Note that this is exactly the transversality condition that you know from the infinite horizon model, and $\beta^T u'(c_T)$ is the proper price for assets in the period of $T + 1$.

Interpretation in the infinite horizon phase diagram we have learned: a path in the phase diagram where $c_t \to 0$ is ruled out by infinite marginal utility, a path where $k = a = -\infty$ is also ruled out by the transversality condition.

(d) Derive the *Consumption* Euler equation of this dynamic system. [5 points]

Solution:

Use the firsrt-order conditions, we have

$$
\begin{aligned}
0 &= \beta^t u'(c_t) - \lambda_t \\
\lambda_t &= \lambda_{t+1}(1 + r_{t+1}),\ t < T \\
&\Rightarrow \beta^t u'(c_t) = \beta^{t+1} u'(c_{t+1})(1 + r_{t+1}) \\
&\Rightarrow u'(c_t) = \beta u'(c_{t+1})(1 + r_{t+1}).
\end{aligned}
$$

Part II: numerical exercise

In Part II you will have to do the calculations with your preferred software (Excel, Matlab, R, or any other software).

(e) Set the parameters of the model to $\beta = 97/100$, $T = 44$, $\theta = 3$, $a_0 = 0$, $r_t = 4/100$, $w_t = 1/2$, and use the Euler equation as well as the period-by-period budget constraint to solve for the optimal consumption and asset accumulation paths.

(Hint: guess a c_0 and then solve the dynamic system forward. If $a_{T+1} > 0$, then you need to increase the initial guess c_0 as leaving assets on the table at the end of your life instead of consuming them cannot be optimal. On the other hand, if $a_{T+1} < 0$, then you need to lower c_0. **You may want to use a solver which is available in most spreadsheet applications to solve for the optimal value c_0 such that $a_{T+1} = 0$ in one step.**[1]).

Using again your preferred software, plot the optimal consumption and asset accumulation path in a **single** diagram with t on the horizontal axis, and label the two paths in the graph clearly. Report also the following quantities (up to the third decimal digit): $c_{t=0}, c_{t=T}$ and $a_{t=1}, a_{t=T}$. [10 points for the graph and 5 points for the reported quantities]

Solution:

For this part, and also for the following parts, see the excel files for specific numbers.

For using the Solver in Excel, roughly it consists of three steps: (1) Load the Solver Add-in (2) Formulate the Model: in this problem, it would be using the Euler equation and the budget constraint, write down the sequence of consumption and assets (starting from $t = 1$) in Excel and these sequences depend on the guessed value of initial consumption, $c_{t=0}$. (3) Solve the Model: We have to find the correct guess for the initial consumption so that savings in the last period (a_{T+1}) is exactly 0.

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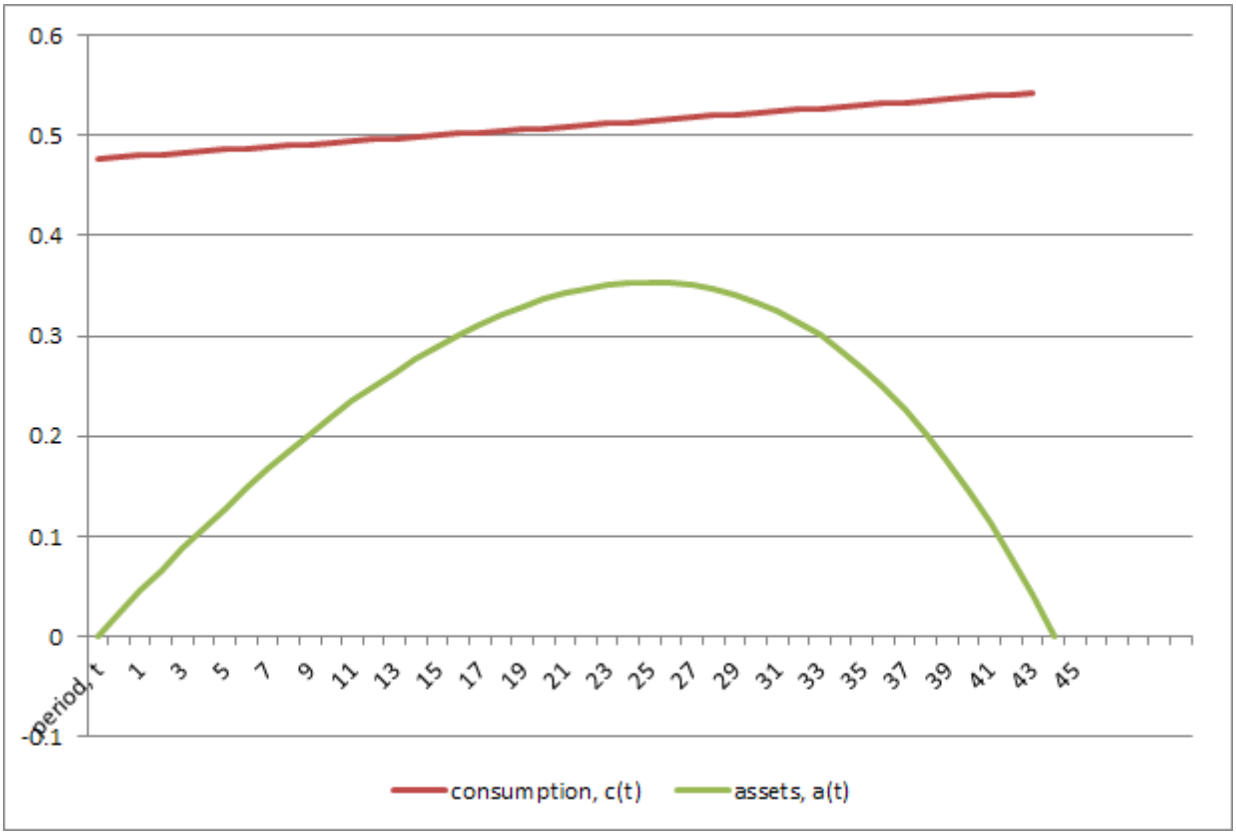
[1] For example some info about the Solver in Excel are available at this link `http://www.excel-easy.com/data-analysis/solver.html`.



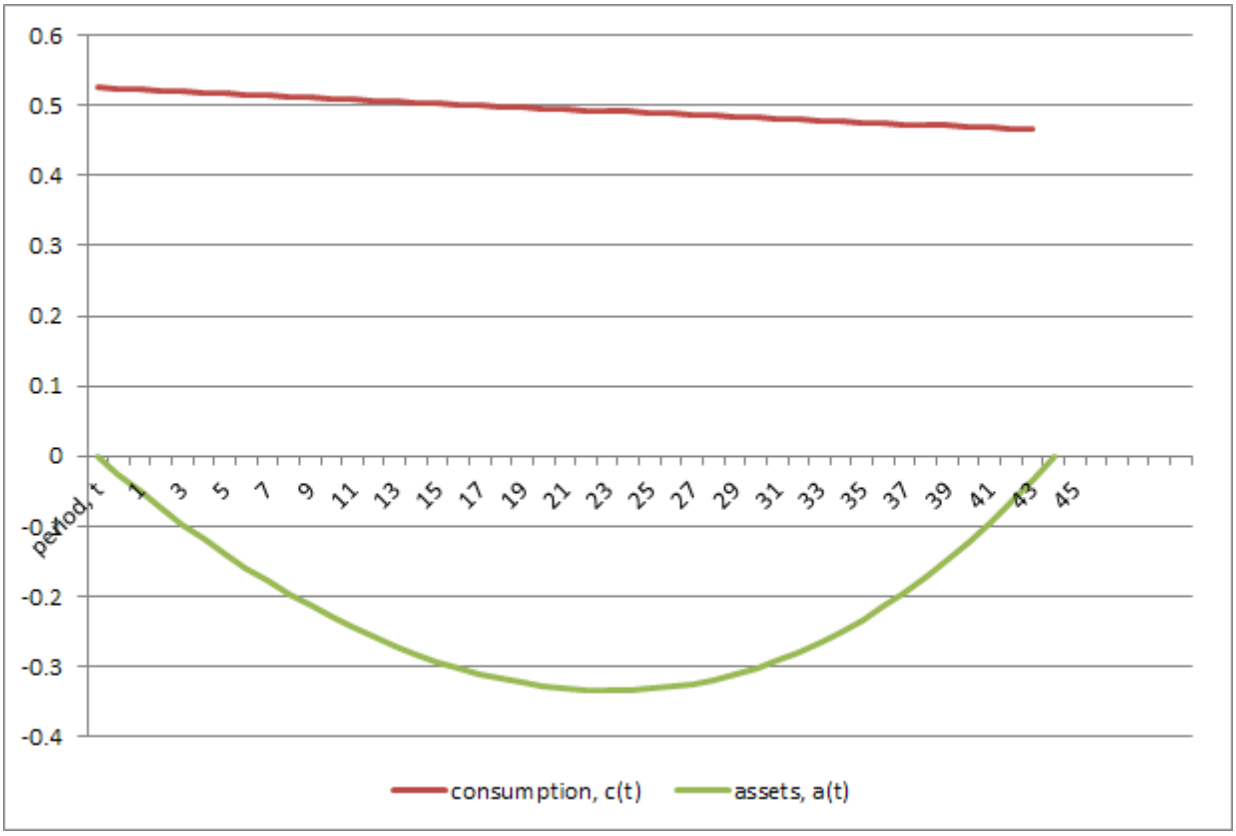
(f) Suppose that $r_t = 2.25/100$ instead of $r_t = 4/100$; all other parameters are the same as in (e). How do the optimal consumption and asset accumulation paths change? Please solve the problem again and plot the new graph for optimal consumption and asset accumulation paths; explicitly point out and comment on the changes. Report also the following quantities (up to the third decimal digit): $c_{t=0}, c_{t=T}$ and $a_{t=1}, a_{t=T}$. [10 points]

Solution:

Main comments: Now $\beta(1+r) < 1$! In (e) you have $\beta(1+r) > 1$. From Consumption Euler equation, you know

$$c_{t+1} = (\beta(1+r))^{1/\theta} c_t$$

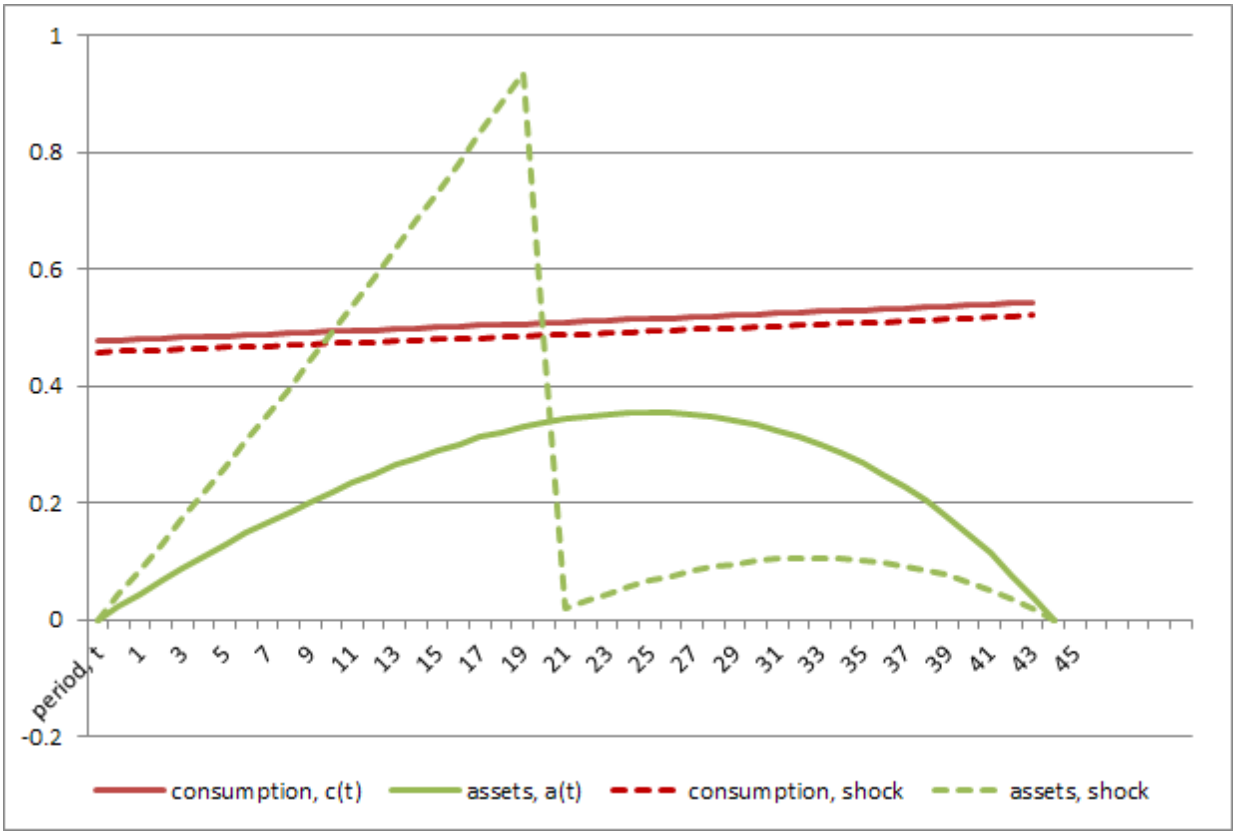
so $(\beta(1+r))^{1/\theta}$ determines the slope of consumption path, or the growth rate of consumptions over time. Consumptions increase over time in (e) while decrease in (f); Consequently, households first accumulate assets and then decumulate assets in (e), while in (f) they first borrow to finance consumption and then repay the debts.



(g) Suppose that $w_{t=20} = 0$ and $w_{t=21} = 0$ (imagine the worker is unemployed for two periods); all other parameters are the same as in (e). How do the optimal consumption and asset accumulation paths change? Please solve the problem again and plot the new graph for optimal consumption and asset accumulation paths. Report also the following quantities (up to the third decimal digit): $c_{t=0}, c_{t=T}$ and $a_{t=1}, a_{t=T}$. Explicitly point out and comment on the changes. [10 points]

Solution:

Main comments: (1) Due to income drop in (g), the consumption levels will shift downward in parallel to that in (e); (2) Consumption paths, both of them are smooth; but that's not the case for assets in (g); (3) Before the income drops, Households clearly save a lot more in (g) relative to (e)(it's known at the beginning; it's not a shock technically).



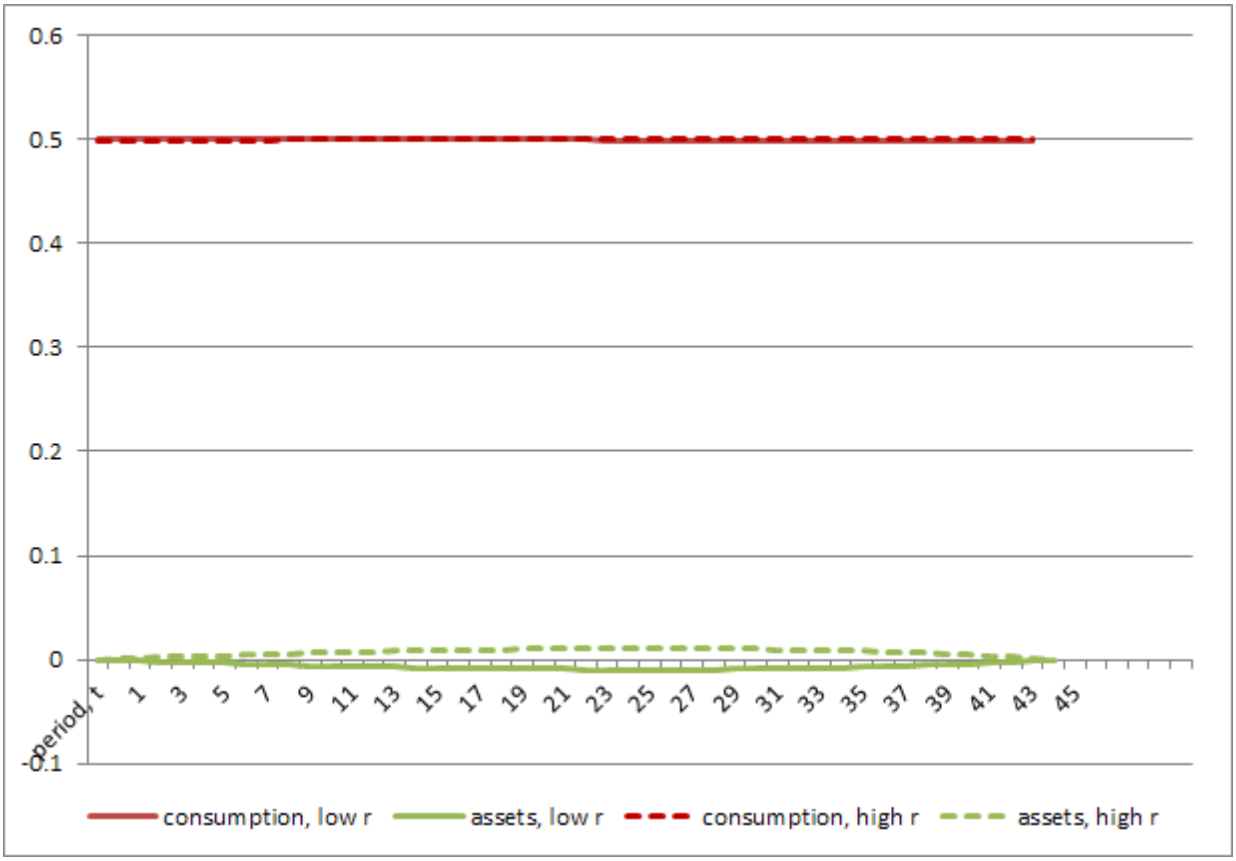
(h) Suppose now that $\theta = 100$. Compare optimal consumption and asset accumulation paths when $r_t = 2.25/100$ and when $r_t = 4/100$. All other parameters are the same as in (e). Please solve the problem again and plot the new graphs for optimal consumption and asset accumulation paths for these two cases; explicitly point out and comment on the differences from point (f). Report also the following quantities for each case (up to the third decimal digit): $c_{t=0}, c_{t=T}$ and $a_{t=1}, a_{t=T}$. [10 points]

Solution:

Main comments: Since θ is extremely large, from Consumption Euler equation,

$$c_{t+1} = (\beta(1+r))^{1/\theta} c_t$$

you can see $(\beta(1+r))^{1/\theta}$ is very close to 1 now. The changes in r will not have big impact on $(\beta(1+r))^{1/\theta}$ and hence the slope of the consumption path. Intuitively, households like very much smoothed consumption path over time (the EIS is $\frac{1}{\theta}$) and interest rate changes will not have big impact on the incentives of adjusting consumption over time.


0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
period, t
1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41 43 45
consumption, low r
assets, low r
consumption, high r
assets, high r