Problem Set 1: Introduction (Solution)

Exercise 1.1: A static competitive equilibrium

Consider a static economy with a representative consumer that has the following preferences over consumption, c, leisure, l, and a public good, g,

$$u(c, l, g) = \log\left(c - \psi(1-l)^{\theta}\right) + \log(g), \quad \theta > 1,$$

and is subject to the budget constraint

$$c = (1-\tau^n)(1-l)w + rk_0.$$

where τ^n is a proportional tax rate on labor income, and k_0 denotes the consumers' initial endowment of physical capital. The representative firm produces consumable output, y, with the following technology

$$y = k^{\alpha}n^{1-\alpha}, \quad 0 < \alpha < 1,$$

by renting physical capital, k, from consumers at the rental rate r, and labor, n, at the wage rate w. The government spends a fixed fraction $\gamma = g/y$ of output on public goods by setting labor income taxes to balance the government's budget

$$\tau^n n w = g.$$

Remember the definition of a competitive equilibrium with a public sector:

> A **competitive equilibrium** is an allocation $\{c, l, k, n\}$ and a set of prices and taxes $\{r, w, \tau^n\}$ such that
>
> (1) The representative consumer chooses c and l to maximize utility subject to the private budget constraint, taking as given prices and taxes.
>
> (2) The representative firm chooses physical capital k and labor input n to maximize profits, taking as given prices.
>
> (3) The government chooses tax policy τ^n to balance the government budget.
>
> (4) The markets for goods, capital, and labor clear.

In what follows, we will compute the competitive equilibrium of this economy step by step.

(a) Solve the consumer's maximization problem.

Solution:

The Lagrangian of the consumer's maximization problem reads

$$\mathcal{L} = \log\left(c - \psi(1-l)^{\theta}\right) + \log(g) + \lambda\left[(1-\tau^n)(1-l)w + rk_0 - c\right].$$

The first-order optimality conditions read

$$0 = \frac{\partial \mathcal{L}}{\partial c} = \frac{1}{c - \psi(1-l)^\theta} - \lambda \tag{1}$$

$$0 = \frac{\partial \mathcal{L}}{\partial l} = \frac{\psi\theta(1-l)^{\theta-1}}{c - \psi(1-l)^\theta} - \lambda(1-\tau^n)w \tag{2}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda} = (1-\tau^n)(1-l)w + rk_0 - c. \tag{3}$$

Substituting out the Lagrange multiplier λ in Equation (2) by using Equation (1) yields the consumers optimal labor supply

$$\psi\theta(1-l)^{\theta-1} = (1-\tau^n)w \quad \Leftrightarrow \quad 1 - l = \left(\frac{\psi\theta}{(1-\tau^n)w}\right)^{1/(1-\theta)}.$$

Using this in the budget constraint (3) yields the optimal consumption level

$$\begin{aligned} c &= (1-\tau^n)w(1-l) + rk_0 \\ &= (1-\tau^n)w(\psi\theta)^{1/(1-\theta)}[(1-\tau^n)w]^{-1/(1-\theta)} + rk_0 \\ &= (\psi\theta)^{1/(1-\theta)}[(1-\tau^n)w]^{-\theta/(1-\theta)} + rk_0. \end{aligned}$$

(b) Derive the optimality conditions of the firm's maximization problem and show that it will make zero profit.
Solution:

The firm maximizes profits π given prices

$$\pi = k^\alpha n^{1-\alpha} - wn - rk.$$

The first-order optimality conditions read

$$0 = \frac{\partial \pi}{\partial n} = (1-\alpha)k^\alpha n^{-\alpha} - w \tag{4}$$

$$0 = \frac{\partial \pi}{\partial k} = \alpha k^{\alpha-1} n^{1-\alpha} - r. \tag{5}$$

Solve the first-order conditions for the prices w and r and compute the associated profits

$$\begin{aligned} \pi &= k^\alpha n^{1-\alpha} - wn - rk \\ &= k^\alpha n^{1-\alpha} - \left[(1-\alpha)k^\alpha n^{-\alpha}\right] n - \left[\alpha k^{\alpha-1} n^{1-\alpha}\right] k \\ &= k^\alpha n^{1-\alpha} - (1-\alpha)k^\alpha n^{1-\alpha} - \alpha k^\alpha n^{1-\alpha} \\ &= 0. \end{aligned}$$

(c) Find the equilibrium prices $\{r, w\}$ that clear the labor and the capital market for a given tax rate τ^n.
Solution:

Clearing in the labor and capital market requires that $k = k_0$ and $n = 1 - l$. Using this in combination with Equation (4) and the optimal labor supply of the agents yields

$$n = 1 - l = \left(\frac{\psi\theta}{(1-\tau^n)(1-\alpha)k_0^\alpha n^{-\alpha}} \right)^{1/(1-\theta)}$$
$$= \left(\frac{\psi\theta}{(1-\tau^n)(1-\alpha)k_0^\alpha} \right)^{1/(1-\theta)} n^{\alpha/(1-\theta)}.$$

Mmultiply both sides by $n^{-\alpha/(1-\theta)}$ to yield

$$n(\tau^n) = \left(\frac{\psi\theta}{(1-\tau^n)(1-\alpha)k_0^\alpha} \right)^{1/(1-\theta-\alpha)}.$$

Using this in Equations (4) and (5) yields the equilibrium prices as a function of the labor income tax

$$w(\tau^n) = (1-\alpha)k_0^\alpha \left(\frac{\psi\theta}{(1-\tau^n)(1-\alpha)k_0^\alpha} \right)^{-\alpha/(1-\theta-\alpha)}$$
$$r(\tau^n) = \alpha k_0^{\alpha-1} \left(\frac{\psi\theta}{(1-\tau^n)(1-\alpha)k_0^\alpha} \right)^{(1-\alpha)/(1-\theta-\alpha)}.$$

(d) Show that the wage income of the consumer is equal to a constant share of output, $wn = (1-\alpha)y$. What is the equilibrium tax rate τ^n of this economy that balances the government budget?
Solution:

We know from the previous analysis that

$$wn = (1-\alpha)k_t^\alpha n^{-\alpha} n$$
$$= (1-\alpha)k_t^\alpha n^{1-\alpha} = (1-\alpha)y,$$

thus the wage income is indeed a constant fraction of output. Alternatively, you could have checked the equality using

$$w(\tau^n)n(\tau^n) = (1-\alpha)k_0^\alpha \left(\frac{\psi\theta}{(1-\tau^n)(1-\alpha)k_0^\alpha} \right)^{(1-\alpha)/(1-\theta-\alpha)}$$
$$= (1-\alpha)k^\alpha n(\tau^n)^{1-\alpha} = (1-\alpha)y(\tau^n).$$

The government budget constraint therefore can be written as

$$\tau^n w(\tau^n)n(\tau^n) = \tau^n(1-\alpha)y(\tau^n) = \gamma y(\tau^n),$$

such that the equilibrium tax rate will be $\tau^n = \gamma/(1-\alpha)$. This tax rate pins down the equilibrium allocation and prices derived above.

(e) Verify that Walras' law holds in the competitive equilibrium derived above, i.e., check whether the goods market clears at equilibrium prices and taxes

$$y(\tau^n) = c(\tau^n) + g.$$

Solution:

Rewrite the market clearing condition as

$$\begin{aligned} y(\tau^n) &= (1-\tau^n)w(\tau^n)n(\tau^n) + r(\tau^n)k + \tau^n w(\tau^n)n(\tau^n) \\ &= w(\tau^n)n(\tau^n) + r(\tau^n)k. \end{aligned}$$

As firms make zero profits, $\pi = 0$, the above equality will indeed be satisfied for any given equilibrium tax rate, τ^n.
