Problem Set 2:
Introduction (Solution)

Exercise 1: Immigration in the Solow model

Consider a closed economy with a neoclassical production function, exogenous technological progress, A_t, a fixed saving rate, s, and a constant labor force, L, as described by the following equations (the Solow model)

$$K_{t+1} - K_t = sY_t - \delta K_t \tag{1}$$
$$Y_t = K_t^\alpha (A_t L)^{1-\alpha},\ 0 < \alpha < 1, \tag{2}$$
$$A_{t+1} = (1+g)A_t,\ A_0 > 0,$$

where $0 \leq \delta \leq 1$ is the depreciation rate of physical capital (in the lecture we have abstracted from depreciation for simplicity).

(a) Remove the trend from Equations (1) and (2) by writing all endogenous variables X_t in terms of efficiency units $x_t \equiv X_t/(A_t L)$.
Solution:

Let us start with aggregate output Y_t. The intensive form of output can be derived by multiplying Equation (2) with $1/(A_t L)$ on both sides

$$y_t = \frac{Y_t}{A_t L} = K_t^\alpha (A_t L)^{-\alpha} = k_t^\alpha.$$

Doing the same multiplication in Equation (1) yields

$$\frac{K_{t+1}}{A_t L} - k_t = sy_t - \delta k_t.$$

As $A_t L = (1+g)^{-1} A_{t+1} L$, the capital accumulation equation can be reformulated as

$$k_{t+1} - k_t = sk_t^\alpha - \delta k_t - gk_{t+1}. \tag{3}$$

(b) Compute the wage rate, w_t, and the rental rate of capital, r_t, (the interest rate of this economy will be $r_t - \delta$) in this economy.
Solution:

The wage rate is given by the marginal product of labor in the production function

$$w_t = \frac{\partial Y_t}{\partial L} = (1-\alpha)K_t^\alpha (A_t L)^{-\alpha} A_t = (1-\alpha)A_t k_t^\alpha \tag{4}$$

and the rental rate is given by the marginal product of capital in the production function

$$r_t = \frac{\partial Y_t}{\partial K_t} = \alpha K_t^{\alpha-1}(A_t L)^{1-\alpha} = \alpha k_t^{\alpha-1}. \tag{5}$$

(c) Compute the stable steady-state capital stock per efficiency unit, $k^\star > 0$, of this economy.
Solution:

We have removed the trend from the dynamic system such that the steady-state condition is given by $k_{t+1} = k_t \equiv k^\star$. Imposing this condition in Equation (3) yields

$$0 = s(k^\star)^\alpha - (\delta + g)k^\star,$$

such that the steady-state capital stock is given by

$$k^\star = \left(\frac{s}{\delta + g}\right)^{1/(1-\alpha)}. \tag{6}$$

Note that there is another steady-state capital stock, $k^\star = 0$, which satisfies the condition. However, this steady-state is not a stable one as the economy would never converge to this capital stock starting from any positive level of capital.

(d) Show that the capital stock per efficiency unit, k_t, is increasing over time as long as $0 < k_t < k^\star$ (hint: look at the ratio k_{t+1}/k_t in the capital accumulation equation to answer this question).
Solution:

Rewrite Equation (3) as

$$k_{t+1} = \frac{1}{1+g}\left[sk_t^\alpha + (1-\delta)k_t\right].$$

The capital stock k_t is increasing over time if $k_{t+1}/k_t > 1$, thus we have to check whether

$$\frac{k_{t+1}}{k_t} = \frac{1}{1+g}\left[sk_t^{\alpha-1} + (1-\delta)\right] > 1,$$

and this equation is equivalent to (note that relation flips as we apply a negative exponential)

$$k_t^{\alpha-1} > \frac{1+g-(1-\delta)}{s} \quad \Leftrightarrow \quad k_t < \left(\frac{s}{\delta+g}\right)^{1/(1-\alpha)} = k^\star.$$

(e) Suppose the economy is in a steady-state. What happens to the aggregate output, the wage rate, and the rental rate on impact if the number of workers in the economy increases by ∂L due to immigration?
Solution:

Output will increase on impact as

$$\frac{\partial Y_t}{\partial L} = (1-\alpha)K_t^\alpha (A_t L)^{-\alpha} A_t = (1-\alpha)A_t k_t^\alpha > 0,$$

the wage rate will drop (the scarcity of labor relative to physical capital decreases)

$$\frac{\partial w_t}{\partial L} = \alpha(1-\alpha)k_t^{\alpha-1}\frac{\partial k_t}{\partial L} < 0,$$

as the capital stock per efficiency unit is decreasing in $\partial k_t/\partial L < 0$. The rental rate on the other hand will increase

$$\frac{\partial r_t}{\partial L} = \alpha(\alpha-1)k_t^{\alpha-2}\frac{\partial k_t}{\partial L} > 0.$$

(f) What happens to aggregate output, the wage rate, and the rental rate over time after the immigration wave?

Solution:

As the capital per efficiency unit, k_t, drops on impact below the steady-state capital stock, $k^\star$, there will be physical capital accumulation after the immigration, $k_{t+1}/k_t > 1$. Consider the following approximation of a variable's net growth rate,

$$g_{X,t} \equiv X_{t+1}/X_t - 1 \approx \log(X_{t+1}/X_t).$$

Then, the growth rate of aggregate output can be approximated by the expression

$$\begin{aligned} g_{Y,t} \approx \log(Y_{t+1}/Y_t) &= \log\left(\frac{A_{t+1}L}{A_tL}\frac{y_{t+1}}{y_t}\right) \\ &= \log(A_{t+1}/A_t) + \log\left((k_{t+1}/k_t)^\alpha\right) \\ &\approx g + \alpha g_{k,t}. \end{aligned}$$

Thus aggregate output will grow above the rate g after immigration (because capital per efficiencey unit grows during the transition, $g_{k,t} > 0$) and then converge back to the steady-state growth rate g. Similarly, the growth rate of wages reads

$$\begin{aligned} g_{w,t} \approx \log(w_{t+1}/w_t) &= \log\left(\frac{(1-\alpha)A_{t+1}}{(1-\alpha)A_t}(k_{t+1}/k_t)^\alpha\right) \\ &= \log(A_{t+1}/A_t) + \alpha\log\left(k_{t+1}/k_t\right) \\ &\approx g + \alpha g_{k,t}, \end{aligned}$$

and thus follows the same growth pattern as output. The growth rate of the rental rate can be approximated by

$$g_{r,t} \approx \log(r_{t+1}/r_t) = \log\left(\frac{\alpha}{\alpha}(k_{t+1}/k_t)^{\alpha-1}\right) = (\alpha-1)g_{k,t},$$

such that the rental rate will fall after the initial jump and converges back to a constant value as the economy converges back to the steady state.

(g) Would the wage rate be higher in the long-run if there had been no immigration?
Solution:

No. In both scenarios the level of technology will be at the same level A_t in every period, and in the long-run the steady-state capital stock per efficiency unit will be the same. Thus, the long-run wage level will be $w_t = (1-\alpha)A_t(k^\star)^\alpha$ under both scenarios.