Problem Set 3:
Ramsey's Growth Model

Exercise 2.1: An infinite horizon problem with perfect foresight

In this exercise we will study at a discrete-time version of Ramsey's growth model. The economy is closed and we consider a representative agent with the following preferences over consumption

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

where c_t denotes period t consumption and $\beta \in (0, 1)$ is the subjective discount factor. The momentary utility function is of the form

$$u(c_t) = \frac{c_t^{1-\theta} - 1}{1 - \theta},$$

with $\theta > 1$. Every period the agent earns a wage w_t (the labor supply is exogenously set to 1 unit), an interest $r_t a_t$ from her assets holdings and she is subject to the lump-sum tax τ_t. In equilibrium, the agent will choose the sequence consumption and asset holdings $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ to maximize U subject to the period-by-period budget constraint

$$c_t + a_{t+1} = w_t + (1 + r_t)a_t - \tau_t, \tag{2}$$

for a given a_0. The agent is atomic and her decisions do not influence aggregate variables, thus she takes the sequence of taxes, wage rates and interest rates as given.

(a) Formulate the Lagrangian of the agent's decision problem (it is common to use λ_t as the Lagrange multiplier on the period t budget constraint). Derive the first-order conditions for the optimal choice of c_t and a_{t+1}, combine these to derive the consumption Euler equation, and give an (micro theory) interpretation of this equation.

(b) Use the Euler equation to show that the functional form of $u(c_t)$ implies a constant elasticity of intertemporal substitution (EIS) between current and future consumption, where

$$\text{EIS} \equiv \frac{\partial \log(c_{t+1}/c_t)}{\partial \log(1 + r_{t+1})}.$$

Give an (consumption growth) interpretation of the EIS.

The representative firm demands physical capital k_t and labor n_t to produce output y_t with the Cobb-Douglas technology

$$y_t = k_t^{\alpha} n_t^{1-\alpha}. \tag{3}$$

The firm is atomic and acts as a price-taking profit maximizer. Capital can be rented at the rental rate $R_t = r_t + \delta$ (note that the depreciation rate δ is the difference between the rental rate and the interest rate) while labor costs w_t.

(c) Find the first-order conditions for the firm's optimization problem.

The government can raise lump-sum taxes τ_t and rolls over debt in the form of one-period bonds, D_{t+1}, to finance government expenditure, G_t. As it pays an interest rate r_t on the outstanding debt, D_t, the government faces a period-by-period budget constraint

$$G_t = \tau_t + D_{t+1} - (1 + r_t)D_t. \tag{4}$$

Moreover, assume that the time path of government debt is such that it is growing at a lower rate than the interest rate

$$\lim_{T\to\infty} \frac{D_{T+1}}{\Pi_{s=0}^{T}(1 + r_s)} = 0.$$

In other words, it is not feasible for the government to finance the outstanding debt (plus interest payments) by issuing ever more debt as time goes by.

(d) Use the government's budget constraint in Equation (4) and substitute for D_t iteratively ($t = 1, 2, 3, ...$) to derive the government's intertemporal budget constraint in net present value (NPV) terms

$$D_0 = \sum_{t=0}^{\infty} \frac{\tau_t - G_t}{\Pi_{s=0}^{t}(1 + r_s)}. \tag{5}$$

Give an interpretation of Equation (5).

(e) Repeating the same procedure for the representative agent's budget constraint in Equation (2) yields the intertemporal private budget constraint in NPV terms

$$a_0 = \sum_{t=0}^{\infty} \frac{c_t + \tau_t - w_t}{\Pi_{s=0}^{t}(1 + r_s)} + \lim_{T\to\infty} \frac{a_{T+1}}{\Pi_{s=0}^{T}(1 + r_s)}$$

What would be the (trivial) solution to the agent's maximization problem if the no-Ponzi condition

$$\lim_{T\to\infty} \frac{a_{T+1}}{\Pi_{s=0}^{T}(1 + r_s)} = 0 \tag{6}$$

was not imposed and assuming that $r_s = r < \infty$?

(f) Assume that Equation (5) holds for a given stream $\{\tau_t, G_t\}_{t=0}^{\infty}$, and so does the no-Ponzi condition in (6). Consider an increase in government expenditures ΔG_t that can be either financed by raising taxes, τ_t, or government debt, D_{t+1}. Does the agent respond differently to a tax-financed relative to a debt financed increase in government expenditures, if she anticipates the government's intertemporal budget constraint? How does your result relate to the Ricardian equivalence proposition?

Remember that the model under consideration is a closed economy and has three markets: the market for labor, the market for consumption goods, and the capital market.

(g) State the three market clearing conditions. Then, solve for the competitive equilibrium variables $\{c_{t+1}, a_{t+1}, k_t, n_t, r_t, w_t, y_t\}_{t=0}^{\infty}$ and the sequence of debt $\{D_{t+1}\}_{t=0}^{\infty}$ as a function of initial consumption c_0, initial assets a_0, initial debt D_0, and the sequence of exogenous government policy $\{G_t, \tau_t\}_{t=0}^{\infty}$ using the first-order conditions, budget constraints and market clearing conditions.