Problem Set 3: Ramsey's Growth Model (Solution)

Exercise 3.1: An infinite horizon problem with perfect foresight

In this exercise we will study at a discrete-time version of Ramsey's growth model. The economy is closed and we consider a representative agent with the following preferences over consumption

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

where c_t denotes period t consumption and $\beta \in (0,1)$ is the subjective discount factor. The momentary utility function is of the form

$$u(c_t) = \frac{c_t^{1-\theta} - 1}{1-\theta},$$

with $\theta > 1$. Every period the agent earns a wage w_t (the labor supply is exogenously set to 1 unit), an interest $r_t a_t$ from her assets holdings and she is subject to the lump-sum tax τ_t. In equilibrium, the agent will choose the sequence consumption and asset holdings $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ to maximize U subject to the period-by-period budget constraint

$$c_t + a_{t+1} = w_t + (1 + r_t)a_t - \tau_t, \tag{2}$$

for a given a_0. The agent is atomic and her decisions do not influence aggregate variables, thus she takes the sequence of taxes, wage rates and interest rates as given.

(a) Formulate the Lagrangian of the agent's decision problem (it is common to use λ_t as the Lagrange multiplier on the period t budget constraint). Derive the first-order conditions for the optimal choice of c_t and a_{t+1}, combine these to derive the consumption Euler equation, and give an (micro theory) interpretation of this equation.

Solution:

The Lagrangian of the constrained optimization problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[(w_t + (1 + r_t)a_t - \tau_t - c_t - a_{t+1}\right]$$

yielding the following first-order conditions for optimal c_t and a_{t+1}

$$0 = \partial\mathcal{L}/\partial c_t = \beta^t u'(c_t) - \lambda_t$$
$$0 = \partial\mathcal{L}/\partial a_{t+1} = -\lambda_t + (1 + r_{t+1})\lambda_{t+1}.$$

Eliminate the Lagrange multiplier λ_t by combining the two and derive the Euler equation for consumption

$$\beta^t u'(c_t) = \beta^{t+1} u'(c_{t+1})(1 + r_{t+1}) \quad \Leftrightarrow \quad \frac{\beta u'(c_{t+1})}{u'(c_t)} = \frac{1}{1 + r_{t+1}}. \tag{3}$$

This is just standard micro theory. The marginal rate of substitution between tomorrow's and today's consumption (the left-hand side) has to be equal to the relative price of tomorrow's in terms of today's consumption (the right-hand side). Relative price interpretation: the agent needs to save $1/(1+r_{t+1})$ units of today's consumption in assets a_{t+1} to yield 1 unit of consumption tomorrow).

(b) Use the Euler equation to show that the functional form of $u(c_t)$ implies a constant elasticity of intertemporal substitution (EIS) between current and future consumption, where

$$\text{EIS} \equiv \frac{\partial \log(c_{t+1}/c_t)}{\partial \log(1+r_{t+1})}.$$

Give an (consumption growth) interpretation of the EIS.

Solution:

First, the marginal utility implied by the functional form of the utility function is

$$u'(c) = c^{-\theta}.$$

The consumption Euler equation then reads

$$\beta \left(\frac{c_{t+1}}{c_t}\right)^{-\theta} = \frac{1}{1+r_{t+1}},$$

take logs on both sides

$$\log \beta - \theta \log \left(\frac{c_{t+1}}{c_t}\right) = -\log(1+r_{t+1}),$$

to yield

$$\text{EIS} = \frac{\partial \log(c_{t+1}/c_t)}{\partial \log(1+r_{t+1})} = 1/\theta.$$

The EIS measures how consumption growth responds to changes in the real interest rate. Consider the case where the EIS is one, $\theta \to 1$. In that case the agent will respond to a one percent increase in the gross interest rate $1+r_{t+1}$, with a one percent increase in consumption growth, c_{t+1}/c_t. On the other hand, if the EIS is very low, $\theta \to \infty$, then the agent would like to keep relative consumption in fixed proportion and does not react to changes in the real interest rate (or, in other words, the consumption smoothing motive is very strong).

The representative firm demands physical capital k_t and labor n_t to produce output y_t with the Cobb-Douglas technology

$$y_t = k_t^{\alpha} n_t^{1-\alpha}. \tag{4}$$

The firm is atomic and acts as a price-taking profit maximizer. Capital can be rented at the rental rate $R_t = r_t + \delta$ (note that the depreciation rate δ is the difference between the rental rate and the interest rate) while labor costs w_t.

(c) Find the first-order conditions for the firm's optimization problem.
Solution:

The firm's optimization problem is static, and it simply maximizes period-by-period profits

$$\pi_t = k_t^\alpha n_t^{1-\alpha} - R_t k_t - w_t n_t.$$

First-order conditions with respect to k_t and n_t are

$$\begin{aligned} 0 &= \partial \pi_t / \partial k_t = \alpha (k_t/n_t)^{\alpha-1} - R_t \\ 0 &= \partial \pi_t / \partial n_t = (1-\alpha)(k_t/n_t)^{\alpha} - w_t, \end{aligned}$$

implying that input factors are paid their marginal product in equilibrium.

The government can raise lump-sum taxes τ_t and rolls over debt in the form of one-period bonds, D_{t+1}, to finance government expenditure, G_t. As it pays an interest rate r_t on the outstanding debt, D_t, the government faces a period-by-period budget constraint

$$G_t = \tau_t + D_{t+1} - (1 + r_t) D_t. \tag{5}$$

Moreover, assume that the time path of government debt is such that it is growing at a lower rate than the interest rate

$$\lim_{T \to \infty} \frac{D_{T+1}}{\Pi_{s=0}^{T}(1 + r_s)} = 0.$$

In other words, it is not feasible for the government to finance the outstanding debt (plus interest payments) by issuing ever more debt as time goes by.

(d) Use the government's budget constraint in Equation (5) and substitute for D_t iteratively ($t = 1, 2, 3, ...$) to derive the government's intertemporal budget constraint in net present value (NPV) terms

$$D_0 = \sum_{t=0}^{\infty} \frac{\tau_t - G_t}{\Pi_{s=0}^{t}(1 + r_s)}. \tag{6}$$

Give an interpretation of Equation (6).
Solution:

Just follow the instructions in the problem. Start out with

$$D_0 = \frac{1}{1 + r_0} \left[\tau_0 - G_0 + D_1 \right].$$

Then insert for D_1 using the same formula

$$\begin{aligned} D_0 &= \frac{1}{1 + r_0} \left[\tau_0 - G_0 + \frac{1}{1 + r_1} \left[\tau_1 - G_1 + D_2 \right] \right] \\ &= \frac{\tau_0 - G_0}{1 + r_0} + \frac{\tau_1 - G_1}{(1 + r_0)(1 + r_1)} + \frac{D_2}{(1 + r_0)(1 + r_1)} \\ &= \sum_{t=0}^{1} \frac{\tau_t - G_t}{\Pi_{s=0}^{t}(1 + r_s)} + \frac{D_{1+1}}{\Pi_{s=0}^{1}(1 + r_s)}, \end{aligned}$$

and continue until period T to get

$$D_0 = \sum_{t=0}^{T} \frac{\tau_t - G_t}{\Pi_{s=0}^{t}(1+r_s)} + \frac{D_{T+1}}{\Pi_{s=0}^{T}(1+r_s)}.$$

Finally, let $T \to \infty$ to yield Equation (6). Thus, the NPV of government expenditures cannot exceed the NPV of lump-sum taxes net of the initial debt position.

(e) Repeating the same procedure for the representative agent's budget constraint in Equation (2) yields the intertemporal private budget constraint in NPV terms

$$a_0 = \sum_{t=0}^{\infty} \frac{c_t + \tau_t - w_t}{\Pi_{s=0}^{t}(1+r_s)} + \lim_{T\to\infty} \frac{a_{T+1}}{\Pi_{s=0}^{T}(1+r_s)}$$

What would be the (trivial) solution to the agent's maximization problem if the no-Ponzi condition

$$\lim_{T\to\infty} \frac{a_{T+1}}{\Pi_{s=0}^{T}(1+r_s)} = 0 \tag{7}$$

was not imposed and assuming that $r_s = r < \infty$?

Solution:

If the no-Ponzi condition was not imposed, then the trivial solution to the constrained optimization problem with a constant interest rate is to issue an unbounded amount of private debt, $a_{t+1} = -\infty$ and enjoy unbounded consumption, $c_t = \infty$. Thus, the no-Ponzi condition makes sure that the NPV of consumption is bounded by the NPV value of wage income net of taxes and initial assets.

(f) Assume that Equation (6) holds for a given stream $\{\tau_t, G_t\}_{t=0}^{\infty}$, and so does the no-Ponzi condition in (7). Consider an increase in government expenditures ΔG_t that can be either financed by raising taxes, τ_t, or government debt, D_{t+1}. Does the agent respond differently to a tax-financed relative to a debt-financed increase in government expenditures, if she anticipates the government's intertemporal budget constraint? How does your result relate to the Ricardian equivalence proposition?

Solution:

Write the government's intertemporal budget constraint as

$$\sum_{t=0}^{\infty} \frac{\tau_t}{\Pi_{s=0}^{t}(1+r_s)} = \sum_{t=0}^{\infty} \frac{G_t}{\Pi_{s=0}^{t}(1+r_s)} + D_0,$$

and use it to express the intertemporal private budget constraint as

$$a_0 = \sum_{t=0}^{\infty} \frac{c_t + \tau_t - w_t}{\Pi_{s=0}^{t}(1+r_s)} = \sum_{t=0}^{\infty} \frac{c_t + G_t - w_t}{\Pi_{s=0}^{t}(1+r_s)} + D_0.$$

Note that the initial outstanding debt D_0 is given and cannot be used by the government to finance the increase in government expenditures.

From the second equality we can see that - once the agent internalizes the intertemporal government budget constraint - it is irrelevant whether G_t is financed with taxes or government debt, as the agent anticipates that issuing government debt today is an NPV equivalent tax liability for tomorrow.

In the economic literature, the proposition that the method of financing government expenditures does not affect private agent's behavior is referred to as the *Ricardian equivalence proposition*. A direct consequence of this proposition is for example that debt-financed tax cuts cannot be used to stimulate consumers' demand for consumption.

Remember that the model under consideration is a closed economy and has three markets: the market for labor, the market for consumption goods, and the capital market.

(g) State the three market clearing conditions. Then, solve for the competitive equilibrium variables $\{c_{t+1}, a_{t+1}, k_t, n_t, r_t, w_t, y_t\}_{t=0}^{\infty}$ and the sequence of debt $\{D_{t+1}\}_{t=0}^{\infty}$ as a function of initial consumption c_0, initial assets a_0, initial debt D_0, and the sequence of exogenous government policy $\{G_t, \tau_t\}_{t=0}^{\infty}$ using the first-order conditions, budget constraints and market clearing conditions.
Solution:

Market clearing for labor requires that all supplied labor is hired (the representative agents supplies one unit of labor)

$$n_t = 1,$$

market clearing in the capital market requires that the agent holds the outstanding government debt and the physical capital in the form of assets

$$a_t = k_t + D_t.$$

By Walras' law we know that market clearing in two markets implies market clearing in the remaining goods market. To see this add the private and government budget constraints in Equation (2) and (5)

$$c_t + G_t + a_{t+1} - D_{t+1} = w_t + (R_t - \delta)(a_t - D_t) + (a_t - D_t) + \tau_t - \tau_t,$$

which is equivalent to

$$c_t + G_t + k_{t+1} = w_t + R_t k_t + (1 - \delta)k_t.$$

Then use the marginal pricing of firms setting $n_t = 1$ to yield the goods market clearing condition

$$\begin{aligned} c_t + G_t + k_{t+1} - (1 - \delta)k_t &= (1 - \alpha)k_t^{\alpha} + \alpha k_t^{\alpha - 1} k_t \\ &= k_t^{\alpha} = y_t, \end{aligned}$$

i.e., the local production y_t is either consumed (private + public) or invested in physical capital.

Given initial consumption c_0 we can now compute the competitive equilibrium variables in an iterative manner. Set $t = 0$, then we can compute all remaining period 0 variables as

$$\begin{aligned} k_0 &= a_0 - D_0 \\ n_0 &= 1 \\ r_0 &= \alpha(a_0 - D_0)^{\alpha-1} - \delta \\ w_0 &= (1-\alpha)(a_0 - D_0)^{\alpha} \\ y_0 &= (a_0 - D_0)^{\alpha}, \end{aligned}$$

and the forward variables as

$$\begin{aligned} a_1 &= w_0 + (1 + r_0)a_0 - \tau_0 - c_0 \\ D_1 &= g_0 - \tau_0 + (1 + r_0)D_0 \\ r_1 &= \alpha(a_1 - D_1)^{\alpha-1} - \delta \\ c_1 &= [\beta(1 + r_1)]^{1/\theta}\, c_0, \end{aligned}$$

where we have used the private budget constraint, the government budget constraint and the consumption Euler equation to compute the latter variables. Apply the same algorithm iteratively for $t = 1, 2, \ldots, \infty$ to compute the whole sequence of equilibrium variables. The equilibrium value of c_0 will then be the level that makes the intertemporal private budget constraint

$$a_0 = \sum_{t=0}^{\infty} \frac{c_t + \tau_t - w_t}{\prod_{s=0}^{t}(1 + r_s)} \left[+ \underbrace{\lim_{T\to\infty} \frac{a_{T+1}}{\prod_{s=0}^{T}(1 + r_s)}}_{=0} \right]$$

holds (or equivalently the no-Ponzi game condition holds). Which is exactly the c_0 that starts the dynamic system on the stable saddle path.