Problem Set 4:
More on Ramsey's Growth Model

Exercise 4.1: Ramsey model in discrete time: a closed form solution

Consider the following version of the Ramsey model with exogenous technology and population growth

$$\max U = \sum_{t=0}^{\infty} \beta^t \log (c_t A_t) L_t, \quad 0 < \beta < 1,$$

subject to

$$\begin{aligned} K_{t+1} &= K_t^{\alpha} (A_t L_t)^{1-\alpha} + (1-\delta) K_t - C_t, \quad 0 < \alpha < 1, \\ A_{t+1} &= (1+g) A_t, \quad A_0 > 0, g > 0, \\ L_{t+1} &= (1+n) L_t, \quad L_0 > 0, n > 0, \end{aligned}$$

where we require that c_t and K_{t+1} remain non-negative, and $K_0 > 0$ is taken as given. The parameter δ denotes the depreciation rate, C_t is aggregate consumption, K_t aggregate physical capital, L_t aggregate labor supply, Y_t aggregate production, A_t labor augmenting productivity, and lower case variables correspond the the same variable in efficiency units $x_t \equiv X_t/(A_t L_t)$.

(a) Remove the trend growth from the capital accumulation equation by restating it in terms of consumption and physical capital per efficiency unit, $c_t \equiv C_t/(A_t L_t)$ and $k_t \equiv K_t/(A_t L_t)$, respectively.
Solution:

Multiply both sides of the capital accumulation equation by the factor $1/(A_t L_t)$ to yield

$$\frac{K_{t+1}}{A_t L_t} = \left(\frac{K_t}{A_t L_t}\right)^{\alpha} + (1-\delta) \frac{K_t}{A_t L_t} - \frac{C_t}{A_t L_t}$$

which can be expressed in terms of efficiency units as

$$k_{t+1}(1+g)(1+n) = k_t^{\alpha} + (1-\delta) k_t - c_t.$$

(b) State the Lagrangian and derive the first-order conditions with respect to consumption c_t and physical capital k_{t+1} per efficiency unit.
Solution:

The Lagrangian of the planner's problem reads

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \log (c_t A_t) L_t + \sum_{t=0}^{\infty} \lambda_t \left[k_t^{\alpha} + (1-\delta) k_t - c_t - k_{t+1}(1+g)(1+n)\right],$$

such that the first-order conditions are given by

$$0 = \frac{\partial \mathcal{L}}{\partial c_t} = \frac{\beta^t A_t L_t}{c_t A_t} - \lambda_t \tag{1}$$

$$0 = \frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\lambda_t(1+g)(1+n) + \lambda_{t+1}\left(\alpha k_{t+1}^{\alpha-1} + 1 - \delta\right) \tag{2}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda_t} = k_t^\alpha + (1-\delta)k_t - c_t - k_{t+1}(1+g)(1+n). \tag{3}$$

(c) Set the depreciation rate of physical capital to $\delta = 1$, and derive the consumption Euler equation. Guess that

$$c_t(k_t) = \gamma k_t^\alpha$$

is the solution to the first-order conditions derived above (this corresponds to the stable saddle-path of the economy). What must be the value of the constant γ in equilibrium? (hint: plug the guess into the Euler equation to derive $k_{t+1}(k_t)$ and then use the capital accumulation equation to determine the constant γ.)

Solution:

To derive the consumption Euler equation combine Equations (1) and (2) and set $\delta = 1$ to yield

$$\frac{\beta^t A_t L_t}{c_t A_t}(1+g)(1+n) = \frac{\beta^{t+1} A_{t+1} L_{t+1}}{c_{t+1} A_{t+1}} \alpha k_{t+1}^{\alpha-1}.$$

As $A_t L_t (1+g)(1+n) = A_{t+1} L_{t+1}$, this equation can be restated in terms of consumption growth

$$\frac{c_{t+1}}{c_t} = \frac{A_t}{A_{t+1}} \beta \alpha k_{t+1}^{\alpha-1} = \frac{\alpha\beta}{1+g} k_{t+1}^{\alpha-1}. \tag{4}$$

As a next step, plug into Equation (4) the guess $c_t(k_t) = \gamma k_t^\alpha$

$$\frac{c_{t+1}}{c_t} = \frac{\gamma k_{t+1}^\alpha}{\gamma k_t^\alpha} = \frac{\alpha\beta}{1+g} k_{t+1}^{\alpha-1} \quad \Leftrightarrow \quad k_{t+1}(k_t) = \frac{\alpha\beta}{1+g} k_t^\alpha.$$

As a final step use this result in Equation (3) to get

$$c_t(k_t) = k_t^\alpha - k_{t+1}(k_t)(1+g)(1+n) = \left[1 - \alpha\beta(1+n)\right] k_t^\alpha,$$

such that $\gamma = 1 - \alpha\beta(1+n)$.

(d) Verify that the solution satisfies the transversality condition

$$\lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = 0.$$

Solution:

Plugging in the solution yields

$$\lim_{t\to\infty} \beta^t c_t^{-1} k_{t+1} = \lim_{t\to\infty} \beta^t \frac{\frac{\alpha\beta}{1+g} k_t^\alpha}{[1-\alpha\beta(1+n)]k_t^\alpha}$$
$$= \frac{\alpha\beta/(1+g)}{1-\alpha\beta(1+n)} \lim_{t\to\infty} \beta^t = 0,$$

because $0 < \beta < 1$.

(e) What is the steady-state physical capital stock per efficiency unit, $k^\star$, in this economy? Sketch the associated phase diagram including the saddle-path with its correct shape.
Solution:

The steady-state capital stock $k^\star$ is given by

$$k^\star = \frac{\alpha\beta}{1+g}(k^\star)^\alpha \quad \Leftrightarrow \quad k^\star = \left(\frac{\alpha\beta}{1+g}\right)^{1/(1-\alpha)}.$$

Sketch phase diagram with a hump-shaped $\dot{k} = 0$ locus, $c_1(k) = k^\alpha - k(1+g)(1+n)$, and a concave-shaped saddle-path, $c_2(k) = \gamma k^\alpha$, such that the steady-state capital stock is lower than the peak of the hump-shaped locus.

Exercise 4.2: An infinite horizon problem with perfect foresight; con't

In this exercise we will study at a discrete-time version of Ramsey's growth model again. The economy is closed and we consider a representative agent with the following preferences over consumption

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{5}$$

where c_t denotes period t consumption and $\beta \in (0, 1)$ is the subjective discount factor. The momentary utility function is of the form

$$u(c_t) = \frac{c_t^{1-\theta} - 1}{1 - \theta},$$

with $\theta > 1$. Every period the agent earns a wage w_t (the labor supply is exogenously set to 1 unit), an interest $r_t a_t$ from her assets holdings and she is subject to the lump-sum tax τ_t. In equilibrium, the agent will choose the sequence consumption and asset holdings $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ to maximize U subject to the period-by-period budget constraint

$$c_t + a_{t+1} = w_t + (1 + r_t)a_t - \tau_t, \tag{6}$$

for a given a_0. The agent is atomic and her decisions do not influence aggregate variables, thus she takes the sequence of taxes, wage rates and interest rates as given.

The representative firm demands physical capital k_t and labor n_t to produce output y_t with the Cobb-Douglas technology

$$y_t = k_t^{\alpha} n_t^{1-\alpha}. \tag{7}$$

The firm is atomic and acts as a price-taking profit maximizer. Capital can be rented at the rental rate $R_t = r_t + \delta$ (note that the depreciation rate δ is the difference between the rental rate and the interest rate) while labor costs w_t.

The government can raise lump-sum taxes τ_t and rolls over debt in the form of one-period bonds, D_{t+1}, to finance government expenditure, G_t. As it pays an interest rate r_t on the outstanding debt, D_t, the government faces a period-by-period budget constraint

$$G_t = \tau_t + D_{t+1} - (1 + r_t)D_t. \tag{8}$$

Moreover, assume that the time path of government debt is such that it is growing at a lower rate than the interest rate

$$\lim_{T\to\infty} \frac{D_{T+1}}{\Pi_{s=0}^{T}(1 + r_s)} = 0.$$

In other words, it is not feasible for the government to finance the outstanding debt (plus interest payments) by issuing ever more debt as time goes by.

The first welfare theorem applies to this economy such that the competitive equilibrium is efficient in the Pareto sense. Thus, we know that the solution to the social planner's problem (which characterizes the Pareto efficient allocation) is equivalent to the competitive market equilibrium. According to the social planner's solution, the same consumption Euler equation and resource constraint (goods market clearing) along with the so-called transversality condition (which stands in for the no-Ponzi condition)

$$\frac{c_{t+1}}{c_t} = [\beta(1+r_{t+1})]^{1/\theta} = \left[\beta(1+\alpha k_{t+1}^{\alpha-1}-\delta)\right]^{1/\theta}$$
$$k_{t+1} - k_t = k_t^\alpha - \delta k_t - c_t - G_t$$
$$\lim_{t\to\infty} \beta^t c_t^{-\theta} k_{t+1} = 0$$

determine the optimal solution of the dynamic system. Let us assume that $G_t = G$, then we can define two correspondances. One which characterizes all possible combinations of (c_t, k_t) when consumption is constant,

$$\mathcal{C}_1(k) \equiv \left\{c \in [0,\infty) : c_{t+1}/c_t = \left[\beta(1+\alpha k^{\alpha-1}-\delta)\right]^{1/\theta}, \ c_{t+1} = c_t = c\right\},$$

and one which captures all combinations if the physical capital stock is constant,

$$\mathcal{C}_2(k) \equiv \{c \in [0,\infty) : c = k_t^a - (k_{t+1} - (1-\delta)k_t) - G, \ k_{t+1} = k_t = k\}.$$

(i) Draw the two correspondances, $\mathcal{C}_1(k)$ and $\mathcal{C}_2(k)$, in a diagram with k on the horizontal axis and c on the vertical axis, the so called phase diagram.
Solution:

See Figure 2.1 in the pdf file for figures.

(j) Comment on the unique point in the phase diagram where the two correspondances intersect.
Solution:

By the construction of $\mathcal{C}_1(k)$ and $\mathcal{C}_2(k)$ the intersection corresponds to the resting point of the dynamic system, the so called steady state where both consumption c_t and physical capital k_t remain constant over time.

(k) Using the phase diagram, illustrate in what direction (c_t, k_t) will move (in all areas of the (c, k)-space).
Solution:

See Figure 2.2 in the pdf file for figures.

(l) Sketch (we do not know the precise shape at this stage) the saddle path leading to the steady state. Explain why any initial consumption off the saddle path cannot be an equilibrium.
Solution:

See Figure 2.3 in the pdf file for figures. Consider any initial level $k_0 > 0$. The saddle path is the unique path that converges to steady state. Remember: we characterized c_0 earlier in the seminar. What if c_0 is chosen too high? Then consumption is ever increasing and physical capital ever falling which violates the positivity restriction. If chosen too low? Then physical capital is ever increasing and consumption ever falling which violates again the positivity restriction.

(m) Consider the steady state consumption level as a function of physical capital

$$c = k^\alpha - \delta k - G.$$

The Golden Rule capital stock is defined as the physical capital stock that maximizes steady-state consumption. Compare the steady state real interest rate of the Ramsey model with the real interest rate that would prevail under the Golden Rule. Is the steady-state physical capital stock in the Ramsey model lower or higher than under the golden rule? Why is that so?

Solution:

The real interest rate can be read from the steady state consumption Euler equation

$$1 = [\beta(1+r)]^{1/\theta},$$

implying that

$$r = \alpha k^{\alpha-1} - \delta = 1/\beta - 1 > 0.$$

On the other hand, the Golden Rule capital stock is characterized by

$$k_{GR} = \arg\max_{k \geq 0} k^\alpha - \delta k - G$$

with the associated first-order condition

$$0 = \alpha k_{\text{GR}}^{\alpha-1} - \delta = r_{\text{GR}}.$$

This implies that the capital stock (the interest rate) in the Ramsey model will be lower (higher) than under the Golden Rule. The simple reason is discounting: in the Ramsey model the planner puts less weight on the long-run (anticipating that accumulating a lot of capital will require consumption sacrifices in the transition), while under the Golden Rule only the long-run outcome counts.