Problem Set 9:
Real Business Cycles (Solution)

Exercise: A two-period real business cycle model

Consider a representative household of a closed economy. The household has a planning horizon of two periods and is endowed with the following preferences over consumption, c, and labor supply, h,

$$U = u(c_1) - v(h_1) + \beta \mathrm{E}\left[u(c_2(s_2)) - v(h_2(s_2))\right],$$

subject to the state-by-state budget constraints

$$\begin{aligned} c_1 + a_2 &= w_1 h_1 \\ c_2(s_2) &= w(s_2)h_2(s_2) + (1+r_2)a_2, \ \forall s_2 \in S \equiv \{s_G, s_B\}. \end{aligned}$$

The variable s_2 denotes the state of the economy in the second period which follows the stochastic process

$$s_2 = \begin{cases} s_G, \text{ with prob. } p \\ s_B, \text{ with prob. } 1-p, \end{cases}$$

and the household conditions the consumption, $c_2(s_2)$, and the labor supply, $h_2(s_2)$, in the second period on the state, s_2. Note that

$$\mathrm{E}[x(s_2)] \equiv px(s_G) + (1-p)x(s_B),$$

denotes the expected value of any variable x that is a function of the future state of the economy, s_2. In each state of the economy, s_t, there is a linear (in labor n_t) production technology of the form

$$y_t(s_t) = A(s_t)n_t(s_t),$$

such that the competitive wage is given by

$$w(s_t) = \frac{\partial y_t(s_t)}{\partial n_t(s_t)} = A(s_t),$$

where the labor productivity is higher in the good state,

$$A(s_G) = A + (1-p)\sigma > A(s_B) = A - p\sigma, \quad \sigma > 0,$$

than in the bad state of the second period. In the first period, the wage is given by

$$w_1 \equiv w(s_1) = \mathrm{E}\left[w(s_2)\right] = A,$$

such that in expectation both periods yield the same wage (labor productivity). Note that σ is a measure of the risk in the economy as

$$\mathrm{Var}[w(s_2)] = \mathrm{E}\left[w(s_2)^2\right] - \mathrm{E}\left[w(s_2)\right]^2 = p(1-p)\sigma^2.$$

Such that the risk in this economy vanishes as $\sigma \to 0$. We will call that case the deterministic economy.

(a) Let $(\lambda_1, \lambda_2(s_G), \lambda_2(s_B))$ denote the Lagrange multipliers of the state-by-state budget constraints. State the representative agent's Lagrangian.
Solution:

By the definition of the expectation operator

$$\begin{aligned} \mathrm{E}\left[u(c_2(s_2)) - v(h_2(s_2))\right] = p\left[u(c_2(s_G)) - v(h_2(s_G))\right] \\ + (1-p)\left[u(c_2(s_B)) - v(h_2(s_B))\right], \end{aligned}$$

such that the Lagrangian can be written in the state-ordered form as

$$\begin{aligned} \mathcal{L} = u(c_1) - v(h_1) + \lambda_1\left[w_1 h_1 - a_2 - c_1\right] \\ + \beta p\left[u(c_2(s_G)) - v(h_2(s_G))\right] + \lambda_2(s_G)\left[w(s_G)h_2(s_G) + (1+r_2)a_2 - c_2(s_G)\right] \\ + \beta(1-p)\left[u(c_2(s_B)) - v(h_2(s_B))\right] + \lambda_2(s_B)\left[w(s_B)h_2(s_B) + (1+r_2)a_2 - c_2(s_B)\right]. \end{aligned}$$

Note that it is important that the return on the asset is not state-contingent. So, technically we are in an incomplete markets environment without aggregate risk (don't tell this the students, this is just an internal notice).

(b) Derive the optimality conditions with respect to consumption, $(c_1, c_2(s_G), c_2(s_B))$, labor supply, $(h_1, h_2(s_G), h_2(s_B))$ and savings, a_2.
Solution:

The optimality conditions with respect to the choices read (we leave away the ones for the Lagrange multipliers)

$$0 = \frac{\partial \mathcal{L}}{\partial c_1} = u'(c_1) - \lambda_1 \tag{1}$$

$$0 = \frac{\partial \mathcal{L}}{\partial c_2(s_G)} = \beta p u'(c_2(s_G)) - \lambda_2(s_G) \tag{2}$$

$$0 = \frac{\partial \mathcal{L}}{\partial c_2(s_B)} = \beta(1-p)u'(c_2(s_B)) - \lambda_2(s_B) \tag{3}$$

$$0 = \frac{\partial \mathcal{L}}{\partial h_1} = -v'(h_1) + \lambda_1 w_1 \tag{4}$$

$$0 = \frac{\partial \mathcal{L}}{\partial h_2(s_G)} = -\beta p v'(h_2(s_G)) + \lambda_2(s_G) w(s_G) \tag{5}$$

$$0 = \frac{\partial \mathcal{L}}{\partial h_2(s_B)} = -\beta(1-p)v'(h_2(s_B)) + \lambda_2(s_B) w(s_B) \tag{6}$$

$$0 = \frac{\partial \mathcal{L}}{\partial a_2} = -\lambda_1 + \left[\lambda_2(s_G) + \lambda_2(s_B)\right](1+r_2). \tag{7}$$

(c) Show that the optimality condition with respect to savings, a_2, can be expressed as the following stochastic consumption Euler equation

$$u'(c_1) = \beta \mathrm{E}\left[u'(c_2(s_2))\right](1+r_2). \tag{8}$$

Then use the intratemporal optimality conditions to show that

$$w_1 u'(c_1) = v'(h_1), \quad w(s_2)u'(c(s_2)) = v'(h(s_2)), \forall s_2 \in S. \tag{9}$$

Solution:

The stochastic consumption Euler equation can be derived by combining Equations (**??**) to (**??**) with Equation (**??**) to eliminate the Lagrange multipliers

$$\begin{aligned} u'(c_1) &= \beta \left[p u'(c_2(s_G)) + (1-p)u'(c_2(s_B)) \right] (1+r_2) \\ &= \beta \mathrm{E} \left[u'(c_2(s_2)) \right] (1+r_2). \end{aligned}$$

Combine Equations (**??**) and (**??**), (**??**) and (**??**), and (**??**) and (**??**), respectively, to yield the intratemporal optimality conditions

$$w_1 u'(c_1) = v'(h_1), \quad w(s_2)u'(c(s_2)) = v'(h(s_2)), \forall s_2 \in S. \tag{10}$$

The intratemporal conditions mean that the marginal cost of working one more unit, $v'(h)$, has to be equal to the value of the increased earnings expressed in terms of marginal utility of consumption. Plugging this into the stochastic consumption Euler equation in Equation (**??**) yields

$$\frac{v'(h_1)}{w_1} = \beta \mathrm{E} \left[\frac{v'(h(s_2))}{w(s_2)} \right] (1+r_2),$$

which is the stochastic Euler equation in terms of the labor supply.

(d) From here onwards we consider the following functional forms for the agent's marginal utility

$$\begin{aligned} u'(c) &= c^{-1}, \text{ (log-utility)} \\ v'(h) &= h^{1/\varphi}, \ \varphi > 0. \end{aligned}$$

Assuming that the asset a_2 is available in zero supply. Show that the equilibrium return on the asset, $1 + r_2$, which clears the capital market (zero asset demand), $a_2 = 0$, is characterized by the equation

$$(1+r_2)\beta = \frac{u'(w_1)}{\mathrm{E}\left[u'(w(s_2))\right]}.$$

Show that marginal utility $u'(w) = w^{-1}$ is a strictly convex function such that Jensen's inequality applies strictly

$$u'(w_1) = u'(\mathrm{E}[w(s_2)]) < \mathrm{E}[u'(w(s_2))],$$

if $w(s_G) \neq w(s_B)$. If $w(s_G) = w(s_B)$, then the above inequality becomes an equality.

Solution:

In equilibrium $a_2 = 0$, such that the state-by-state budget constraints imply the following consumption levels

$$c_1 = w_1 h_1$$
$$c_2(s_2) = w(s_2)h_2(s_2),\ \forall s_2 \in S.$$

The intertemporal optimality conditions imply

$$w_1 c_1^{-1} = h_1^{1/\varphi} \quad \Leftrightarrow \quad h_1 = \left[w_1 c_1^{-1}\right]^{\varphi} = \left[w_1 (w_1 h_1)^{-1}\right]^{\varphi} = h_1^{-\varphi},$$

such that the optimal labor supply is given by

$$h_1 = 1.$$

The same transformation holds for the optimal labor supply in the second period

$$h_2(s_2) = 1,$$

such that optimal consumption is given by

$$c_1 = w_1 h_1 = w_1.$$
$$c_2(s_2) = w(s_2)h_2(s_2) = w(s_2),\ \forall s_2 \in S.$$

Use this in the stochastic consumption Euler equation in Equation (**??**) to yield

$$(1 + r_2)\beta = \frac{u'(w_1)}{\mathrm{E}\left[u'(w(s_2))\right]} = \frac{u'(\mathrm{E}[w(s_2)])}{\mathrm{E}\left[u'(w(s_2))\right]}.$$

Since the marginal utility function can be shown to be strictly convex (marginal utility is strictly convex when the third derivate of the utility function is strictly positive. This is the necessary condition to have precautionary savings)

$$u'(w) \equiv w^{-1}$$
$$u''(w) = -1w^{-2} < 0$$
$$u'''(w) = 2w^{-3} > 0,$$

Jensen's inequality can be applied to this ratio in the strict sense, if $\sigma > 0$ as then $w(s_G) \neq w(s_B)$. (INTERNAL NOTE: ILLUSTRATE JENSEN'S' INEQUALITY IN A DIAGRAM FOR THE CONVEX MARGINAL UTILITY.)

(e) Compare the equilibrium labor supply in the first period, h_1 and the interest rate, $1 + r_2$, of the stochastic economy ($\sigma > 0$) to the equilibrium variables in the deterministic economy ($\sigma \to 0$). What is your conclusion, how do the optimal labor supply and the interest rate respond to an increase in risk, σ?
Solution:

In the stochastic economy with $\sigma > 0$, Jensen's inequality implies

$$(1+r_2)\beta = \frac{u'(w_1)}{\mathrm{E}\left[u'(w(s_2))\right]} = \frac{u'(\mathrm{E}\left[w(s_2)\right])}{\mathrm{E}\left[u'(w(s_2))\right]} < 1,$$

such that the interest rate in this economy is smaller than the inverse of the discount rate, $(1+r_2) < 1/\beta$. In the deterministic economy, the consumption Euler equation implies

$$(1+r_2)\beta = \frac{u'(\mathrm{E}\left[w(s_2)\right])}{\mathrm{E}\left[u'(w(s_2))\right]} = \frac{u'(A)}{pu'(A)+(1-p)u'(A)} = 1,$$

such that the interest rate is equal to the inverse of the discount factor, $(1+r_2) = 1/\beta$ and therefore higher than in the stochastic economy. (INTERNAL NOTE: ILLUSTRATE THIS IN A DIAGRAM WITH a_2 ON THE VERTICAL AND $1+r_2$ ON THE HORIZONTAL AXIS. ASSET SUPPLY IS THEN HORIZONTAL, ASSET DEMAND STRICTLY INCREASING IN $1+r_2$.)

The reason why the interest rate is lower in the stochastic case is that the demand for assets is higher because of the precautionary savings motive of the household, such that the return to savings has to be lower to clear the market given the same zero asst supply.

The optimal labor supply in period one on the other hand is the same across the two models,

$$h_1 = 1,$$

and is unaffected by an increase in the risk of future labor productivity.

(f) In parts (d) and (e) we have assumed that the asset is given in zero supply. Here we relax this assumption and assume instead that the household faces an exogenous interest rate

$$1+r_2 = 1/\beta,$$

but may choose nonzero savings at this interest rate. We already know from parts (d) and (e) that in the deterministic economy ($\sigma \to 0$) this interest rate implies zero asset holdings, $a_2 = 0$, and a constant labor supply across periods

$$h_1 = h_2 = 1.$$

Now, let us see what is the optimal labor supply, h_1, and savings, a_2, in the stochastic economy. Suppose we knew the optimal level of savings in the stochastic economy and let denote this level by $\tilde{a}_2$. Show first that the optimal labor supply in the first period, h_1, is increasing in the level of savings, $\tilde{a}_2$. Then show that the optimal level of savings will be strictly positive, $\tilde{a}_2 > 0$. Thus, what is your conclusion about how the optimal labor supply in the first period responds to an increase in risk, σ?

Solution:

From the budget constraint we know that consumption in period zero is given by

$$c_1 = w_1 h_1 - \tilde{a}_2,$$

The intratemporal optimality condition then implies

$$h_1 = w_1^{\varphi} c_1^{-\varphi} = w_1^{\varphi}(w_1 h_1 - \tilde{a}_2)^{-\varphi}.$$

Let's define implicitly the optimal labor supply as a function of $\tilde{a}_2$

$$G(h_1(\tilde{a}_2), \tilde{a}_2) \equiv w_1^{\varphi}(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-\varphi} - h_1(\tilde{a}_2) = 0,$$

such that the implicit function theorem gives us the response of the first period labor supply with respect to the savings $\tilde{a}_2$ as (note that d denotes total derivatives and ∂ partial derivatives)

$$\frac{dG(h_1(\tilde{a}_2), \tilde{a}_2)}{d\tilde{a}_2} = \frac{\partial G(h_1(\tilde{a}_2), \tilde{a}_2)}{\partial h_1(\tilde{a}_2)} \frac{dh_1(\tilde{a}_2)}{d\tilde{a}_2} + \frac{\partial G(h_1(\tilde{a}_2), \tilde{a}_2)}{\partial \tilde{a}_2} = 0,$$

such that

$$\begin{aligned}
\frac{dh_1(\tilde{a}_2)}{d\tilde{a}_2} &= -\frac{\partial G(h_1(\tilde{a}_2), \tilde{a}_2)/\partial \tilde{a}_2}{\partial G(h_1(\tilde{a}_2), \tilde{a}_2)/\partial h_1(\tilde{a}_2)} \\
&= -\frac{w_1^{\varphi}(-\varphi)(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-\varphi-1}(-1)}{w_1^{\varphi}(-\varphi)(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-\varphi-1} w_1 - 1} \\
&= \frac{\varphi w_1^{\varphi}(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-\varphi-1}}{\varphi w_1^{\varphi}(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-\varphi-1} w_1 + 1} \\
&= \frac{\varphi w_1^{\varphi}(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-\varphi}}{\varphi w_1^{\varphi}(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-\varphi} w_1 + (w_1 h_1(\tilde{a}_2) - \tilde{a}_2)} \frac{(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-1}}{(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-1}} \\
&= \frac{\varphi h_1(\tilde{a}_2)}{\varphi h_1(\tilde{a}_2) w_1 + c_1(\tilde{a}_2)} > 0,
\end{aligned}$$

such that household will increase the labor supply in the first period with the level of savings. Note that this relationship between the labor supply and savings is valid independent on whether there is risk in the economy and applies to both cases where the asset demand is allowed to be different from zero in equilibrium.

Can $\tilde{a}_2 = 0$, $h_1 = h_2(s_2) = 1$, be the equilibrium of the stochastic economy? No, the following Euler equation has to be satisfied at the optimal level of savings, $\tilde{a}_2$,

$$\beta(1 + r_2) = 1 = \frac{u'(c_1)}{\mathrm{E}\left[u'(c_2(s_2))\right]} = \frac{u'(w_1 h_1 - \tilde{a}_2)}{\mathrm{E}\left[u'(w(s_2) h_2(s_2) + (1 + r_2)\tilde{a}_2)\right]},$$

and we have already seen that zero asset holdings are only consistent with an interest rate smaller than $1/\beta$ in the stochastic economy

$$\frac{u'(w_1 h_1 - \tilde{a}_2)}{\mathrm{E}\left[u'(w(s_2) h_2(s_2) + (1 + r_2)\tilde{a}_2)\right]} = \frac{u'(w_1)}{\mathrm{E}\left[u'(w(s_2))\right]} < 1 = \beta(1 + r_2),$$

by Jensen's inequality. So, in the stochastic equilibrium the marginal utility in the first-period has to be higher (first-period consumption has to be lower) than in the deterministic equilibrium and this is only possible by saving a positive amount, $\tilde{a}_2 > 0$. This also increases the labor supply in the first period $h_1 > 1$ (but only to the extent that consumption is lower and marginal utility higher than before).

In summary, with an exogenously given interest rate $(1 + r_2) = 1/\beta$, the household saves more (precautionary savings) and works more in the first period as the risk in the economy increases. (INTERNAL NOTE: ILLUSTRATE THIS IN A DIAGRAM WITH a_2 ON THE VERTICAL AND $1 + r_2$ ON THE HORIZONTAL AXIS. ASSET SUPPLY IS THEN VERTICAL, ASSET DEMAND STRICTLY INCREASING IN $1 + r_2$.)

(g) As a last step of the analysis, consider a closed economy where the demand for assets must be equal to the demand for physical capital in the second period

$$a_2 = k_2(r_2).$$

Thus, the demand for capital, k_2 is a decreasing function of the endogenous interest rate, r_2, (think of the interest rate as the marginal product of capital, the higher the capital the lower the marginal product of capital). Intuitively, what will be the response of first-period labor supply, savings, and the capital in this economy if the risk σ increases?

Solution:

The mechanism is very similar to the cases analyzed before: as the risk in the economy increases, the precautionary savings motive of the household increases. Would the interest rate stay unchanged, then the demand for assets exceeds the demand for capital which can not be a capital market equilibrium. Thus, the interest rate will have to fall and this reduces on the one hand the demand for assets and on the other hand it increases the demand for capital (the asset supply!) until the capital market clears again.

The difference to the previous case is that the interest rate will fall less (be higher) compared to the case where the asset supply was fixed, as also the available capital is expanding with the falling interest rate. Also, the increase of the labor supply in the first period will be less pronounced in the first-period, but increase with the amount of savings. (INTERNAL NOTE: ILLUSTRATE THIS IN A DIAGRAM WITH a_2 ON THE VERTICAL AND $1 + r_2$ ON THE HORIZONTAL AXIS. ASSET SUPPLY IS THEN STRICTLY DECREASING, ASSET DEMAND STRICTLY INCREASING IN $1 + r_2$.)

Exercise : The Laffer Curve

Consider a representative household of a static economy with the following preferences over private consumption, c, labor supply, h, and public goods, g

$$U = \max \left[\frac{\left(c - \frac{h^{1+1/\varphi}}{1+1/\varphi}\right)^{1-\theta} - 1}{1-\theta} + \sigma \log(g) \right], \quad \theta > 0, \tag{11}$$

where $0 < \varphi < \infty$ denotes the Frisch elasticity of labor supply, and $\sigma > 0$ is a parameter that measures the household's relative preference for public over private goods. The household faces an exogenously given wage rate, w, and interest rate, r, and labor income is taxed at the proportional rate τ^n yielding the private budget constraint (the household is born without assets)

$$c = (1 - \tau^n) w h. \tag{12}$$

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(a) Write down the household's optimality conditions with respect to consumption, c and labor supply, h (the public good provision by the government is taken as given), and derive the optimal labor supply which we will denote by $h(\tau^n)$.
Solution:

The households maximizes utility subject to Equation (**??**), such that the Lagrangian reads

$$\mathcal{L} = \frac{\left(c - \frac{h^{1+1/\varphi}}{1+1/\varphi}\right)^{1-\theta} - 1}{1-\theta} + \sigma \log(g) + \lambda \left[(1 - \tau^n) w h - c\right].$$

The optimality conditions with respect to consumption c and labor supply, h, are given by

$$0 = \frac{\partial \mathcal{L}}{\partial c} = \left(c - \frac{h^{1+1/\varphi}}{1+1/\varphi}\right)^{-\theta} - \lambda$$

$$0 = \frac{\partial \mathcal{L}}{\partial h} = -\left(c - \frac{h^{1+1/\varphi}}{1+1/\varphi}\right)^{-\theta} h^{1/\varphi} + \lambda (1 - \tau^n) w.$$

Combining the two optimality conditions and substituting for λ yields

$$\left(c - \frac{h^{1+1/\varphi}}{1+1/\varphi}\right)^{-\theta} h^{1/\varphi} = \left(c - \frac{h^{1+1/\varphi}}{1+1/\varphi}\right)^{-\theta} (1 - \tau^n) w,$$

which can be simplified to find the optimal labor supply

$$h(\tau^n) = \left[(1 - \tau^n) w\right]^{\varphi}. \tag{13}$$

(b) Compute the elasticity of the labor supply with respect to the tax rate

$$e(\tau^n) \equiv -\frac{\partial h(\tau^n)}{\partial \tau^n}\frac{\tau^n}{h(\tau^n)}.$$

Show that this elasticity is increasing in the tax rate, τ^n, i.e., the higher the tax rate the more distorted is the labor supply in this economy.
Solution:

The labor supply elasticity with respect to the tax rate is given by

$$e(\tau^n) = -\varphi\left[(1-\tau^n)w\right]^{\varphi-1}(-w)\frac{\tau^n}{\left[(1-\tau^n)w\right]^{\varphi}} = \varphi\frac{\tau^n}{1-\tau^n}.$$

Note that the elasticity (and therefore the tax distortion on the labor supply) is monotonically increasing in the tax rate on the interval $\tau_t \in [0,1)$,

$$\frac{\partial e(\tau^n)}{\partial \tau^n} = \varphi\left[\frac{1}{1-\tau^n} + \frac{\tau^n}{(1-\tau^n)^2}\right] = \varphi\frac{1}{(1-\tau^n)^2} > 0,$$

where $e(0) = 0$ and $\lim_{\tau^n \to 1} e(\tau^n) = \infty$.

(c) Derive the government's labor income tax revenue as a function of the tax rate - the so called Laffer curve. What tax rate $\bar{\tau}$ is associated with the top of the Laffer curve (the maximum tax revenue)? What value takes the elasticity $e(\tau)$ at the top of the Laffer curve? What was the tax rate at the top of the Laffer curve if the labor supply is completely inelastic, $\varphi \to 0$, or inelastic, $\varphi \to \infty$?
Solution:

(XX Reminder: put up a .pptx figure with the Laffer curve for next year) The tax revenue of the government is given by

$$\tau^n w h(\tau^n) = \tau^n w\left[(1-\tau^n)w\right]^{\varphi},$$

such that the top of the Laffer curve is characterized by

$$\bar{\tau} = \arg\max_{0\le\tau^n\le 1} \tau^n w\left[(1-\tau^n)w\right]^{\varphi}.$$

The first-order condition reads

$$\begin{aligned} 0 &= w\left[(1-\bar{\tau})w\right]^{\varphi} - \varphi\bar{\tau}w\left[(1-\bar{\tau})w\right]^{\varphi-1}w \\ &= w\left[(1-\bar{\tau})w)\right]^{\varphi}\left[1-\varphi\bar{\tau}(1-\bar{\tau})^{-1}\right] = w\left[(1-\bar{\tau})w)\right]^{\varphi}\left[1-e(\bar{\tau})\right]. \end{aligned}$$

The first-order condition suggests $\bar{\tau} = 1$ as one of the candidate solutions. However, since $\bar{\tau} = 1$ implies a tax revenue of zero we can safely drop it as a maximum candidate. Due to the ruling out the unit tax rate, we can divide both sides of the above optimality condition by $w\left[(1-\bar{\tau})w\right]^{\varphi}$ to characterize the tax rate that maximizes tax revenue

$$1 = \varphi\bar{\tau}(1-\bar{\tau})^{-1} = e(\bar{\tau}) \quad \Leftrightarrow \quad \bar{\tau} = \frac{1}{1+\varphi}.$$

The above equation implies that the elasticity at the top of the Laffer curve is exactly 1. The two limit cases yield

$$\lim_{\varphi\to 0} \bar{\tau} = 1, \quad \lim_{\varphi\to\infty} \bar{\tau} = 0,$$

Thus, the less elastic the labor supply is, the higher is the maximum revenue that the government can generate from taxing labor income. Note that the case of lump-sum taxation is nested when $\varphi \to 0$, such that the government generates the maximum tax revenue at the unit tax rate.

Note that $\bar{\tau} = 1/(1+\varphi)$ is indeed a maximizer as the tax revenue is globally concave

$$\begin{aligned}
\frac{\partial[\tau^n wh(\tau)]}{\partial\tau^n} &= w\left[(1-\tau^n)w)\right]^{\varphi}\left[1-e(\tau^n)\right] \\
\frac{\partial^2[\tau^n wh(\tau)]}{\partial\tau^n\partial\tau^n} &= \varphi w\left[(1-\tau^n)w)\right]^{\varphi-1}(-w)\left[1-e(\tau^n)\right] \\
&\quad + w\left[(1-\tau^n)w)\right]^{\varphi}(-1)\frac{\partial e(\tau^n)}{\partial\tau^n} < 0.
\end{aligned}$$

(d) Suppose the government wants to finance the specific level of government expenditure $g^\star$ that is located within the bounds

$$0 < g^\star < \bar{\tau}wh(\bar{\tau}).$$

Assume that $\varphi = 1$. Find the optimal tax rate, $\tau^\star$, to finance the government expenditure level, $g^\star$, with a balanced government budget. Would a benevolent government ever choose a tax rate above $\bar{\tau}$?

Solution:

When $\varphi = 1$ then the top of the Laffere curve is given by $\bar{\tau} = 1/2$ and the maximum tax revenue is

$$\bar{\tau}w\left[(1-\bar{\tau})w\right] = 1/4w^2 > g^\star.$$

The government budget constraint reads

$$\begin{aligned}
g^\star &= \tau^\star wh(\tau^\star) \\
&= \tau^\star w\left[(1-\tau^\star)w\right] = \tau^\star w^2 - (\tau^\star)^2 w^2,
\end{aligned}$$

which can be written as the quadratic equation

$$\begin{aligned}
0 &= w^2(\tau^\star)^2 - w^2\tau^\star + g^\star \\
&\equiv ax^2 + bx + c.
\end{aligned}$$

The two solutions are characterized by

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
&= \frac{w^2 \pm \sqrt{w^4 - 4w^2 g^\star}}{2w^2} = 1/2 \pm \frac{\sqrt{1-4w^{-2}g^\star}}{2}.
\end{aligned}$$

Since it can never be optimal to raise the same tax revenue $g^\star$ with a higher tax rate than necessary (because it would increase the labor supply distortion and reduce the available budget of the household unnecessarily), the optimal tax rate must be given by

$$\tau^\star = 1/2 - \frac{\sqrt{1 - 4w^{-2}g^\star}}{2} < \bar{\tau} = 1/2.$$

Note that the term under the square root is strictly positive since $g^\star < 1/4w^2$.