Problem set 1

ECON 4330

Part 1

We are looking at an open economy that exists for two periods. Output in each period Y_1 and Y_2 respectively, is given exogenously. A representative consumer maximizes life-time utility

$$U = u(C_1) + \beta u(C_2)$$

where C_1 and C_2 are consumption in the two periods and β is a subjective discount factor, $0 < \beta < 1$.The country can borrow and lend in world markets at a given real interest rate, r. The initial asset is zero. Hence, the budget constraint can be written

$$C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r}$$

1. *Derive the first order condition for optimal consumption and interpret it.*

   solution Using the method of substitution (if you want, use Lagrange method). Solve the budget constraint for C_2

   $$C_2 = Y_1(1+r) + Y_2 - (1+r)C_1 \tag{1}$$

   and substitute the constraint into the objective function

   $$U = u(C_1) + \beta u(Y_1(1+r) + Y_2 - (1+r)C_1) \tag{2}$$

   Maximize the objective function wrt C_1. The FOC is

   $$\begin{aligned} u'(C_1) - \beta u'(C_2)(1+r) &= 0 \\ &\Leftrightarrow \\ u'(C_1)\frac{1}{1+r} &= \beta u'(C_2) \\ &\Leftrightarrow \\ \frac{\beta u'(C_2)}{u'(C_1)} &= \frac{1}{1+r} \end{aligned} \tag{3}$$

   The left-hand side of the last expression is the marginal rate of substituion (MRS) between consumption in period 1 and 2, i.e. how much

C_1 you are willing to trade for one more unit C_2 (given constant utility). The right hand side is the relative price of consumption in period 2, i.e. how much C_1 you have to give up to get one more unit C_2. In optimum $MRS(C_1, C_2)$=relative price. If $MRS > \frac{1}{1+r}$ you can increase utility by trading C_1 for C_2 and vice versa if $MRS < \frac{1}{1+r}$.

2. *Derive the welfare effects of an increase in r, i.e dU/dr. Provide intuition. (use the envelope condition)*

solution Substituting 1 into 3 , gives us one equataion to the determine optimal period 1 consumption C_1^* as function of the exogenous variables $C_1^*(r, Y_1, Y_2)$. Optimal period two consumption follows from eq. 1: $C_2^*(r, Y_1, Y_2) = (1+r)Y_1 + Y_2 - (1+r)C_1^*(r, Y_1, Y_2)$. Substituting this into the objective function gives the *indirect utility function,* i.e. maximal utility as a function of exogenous variables

$$U^*(r, Y_1, Y_2) = u(C_1^*(r, Y_1, Y_2)) + \beta u(Y_1(1+r) + Y_2 - (1+r)C_1^*(r, Y_1, Y_2))$$

To see how welfare is affected by an increase in the interest rate we derive

$$\begin{aligned} \partial U^*/\partial r &= u'(C_1^*)\frac{\partial C_1^*}{\partial r} + \beta u'(C_2^*)\left[Y_1 - C_1^* - (1+r)\frac{\partial C_1^*}{\partial r}\right] \\ &= \beta u'(C_2^*)\,(Y_1 - C_1^*) + \frac{\partial C_1^*}{\partial r}\left[u'(C_1^*) - (1+r)\beta u'(C_2^*)\right] \end{aligned}$$

In optimum the FOC holds, $u'(C_1^*) - (1+r)\beta u'(C_2^*) = 0$, and we are left with

$$\partial U^*/\partial r = \beta u'(C_2^*)\,(Y_1 - C_1^*)$$

We the see that the sign of the welfare effect depends on the sign of $(Y_1 - C_1^*)$, that is whether the country is a borrower or a lender in the first period. If the country is a lender in the first period $Y_1 > C_1^*$, welfare improves when the interest rate increase. Note that an increase in the interest rate is the same as a reduction in the relative price of period 2 consumption. If the country is a lender, it exports period 1 consumption and imports period 2 consumption. A reduction in the interest rate therefore improves the country's terms of trade (the price of imports)/(price of exports). It now gets more for what it exports and pay less for what it imports, and the country is better off.

3. *Assume CRRA utility*

$$u(C) = \frac{C^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$$

Find an expression for date 1 consumption and current account as functions of exogenous variables (Y_1, Y_2, r).

solution Use:

$$C_2 = (\beta(1+r))^\sigma C_1$$

and the budget constraint

$$\begin{aligned} C_1 + \frac{C_2}{1+r} &= Y_1 + \frac{Y_2}{1+r} \\ C_1(1+\beta^\sigma(1+r)^{\sigma-1}) &= Y_1 + \frac{Y_2}{1+r} \\ C_1 &= \frac{1}{(1+\beta^\sigma(1+r)^{\sigma-1})}(Y_1 + \frac{Y_2}{1+r}) \end{aligned}$$

4. *Assume* $Y_2 = 0$

(a) *Derive* $\partial C_1/\partial r$ *and find condition that makes sure the current account is improving when the world interest rate goes up.*

solution Current account in a model without investment $CA_1 = S_1 = Y_1 - C_1$. To find the consumption response to r use the expression from the last problem with $Y_2 = 0$

and the budget constraint

$$C_1 = \frac{1}{(1+\beta^\sigma(1+r)^{\sigma-1})}Y_1$$

take the derivative wrt r to get

$$\frac{\partial C_1}{\partial r} = -\frac{(\sigma-1)\,\beta^\sigma(1+r)^{\sigma-2}}{(1+\beta^\sigma(1+r)^{\sigma-1})^2}Y_1$$

If $\sigma > 1$ the date 1 consumption goes down and saving increases. Intuition: High σ implies that the substitution effect is large!

(b) *Describe how* C_1 *responds to changes in* r *in terms of substitution and income effects.*

solution Let $p_2 = \frac{1}{1+r}$. An increase in r reduces the price of C_2. The agent therefore substitutes towards C_2, the good that is now relatively cheaper. But as p_2 goes down the agent pays less for period 2 consumption, which increases real income. The increase in income increases both C_1 and C_2. Hence, for C_1 there is a negative substitution effect and a positive income effect. If $\sigma > 1$ the substitution effect dominates. If $\sigma < 1$ the income effect dominates. If $\sigma = 1$ (log utility) the income and substitution effects exactly offset each other, and C_1 does not change.

(c) *What additional effect comes in if we assume* $Y_2 > 0$*?*

solution with a positive Y_2 a decrease in p_2 reduces the value of period 2 endowment. Hence, there is an additional negative wealth effect that reduces both C_1 and C_2. Note that the income effect is given by

$$\frac{\partial C_1}{\partial I}C_2$$

where $I = Y_1 + p_2 Y_2$. The wealth effect is given by

$$-\frac{\partial C_1}{\partial I}\frac{\partial I}{\partial p_2} = -\frac{\partial C_1}{\partial I}Y_2$$

Hence the combined income and wealth effect is

$$(C_2 - Y_2)\frac{\partial C_1}{\partial I}$$

5. *Suppose a foreign country has the same preferences as the home country, equal date 1 output $Y_1^* = Y_1$ but different date 2 output Y_2^**

(a) *Assume higher income growth in the home country, i.e. $Y_2 > Y_2^*$. Derive the autarky interest rate in both countries and compare.*

solution The autarky interest rate r_A is the interest rate that induces the agent to set consumption equal to income in both periods (the only possible allocation in a closed economy, as we don't have real capital in our model). With CRRA utility the FOC in autarky is

$$\frac{\beta C_2^{-\frac{1}{\sigma}}}{C_1^{-\frac{1}{\sigma}}} = \frac{1}{1+r_A}$$

We find the autarky interest rate by inserting for the only possible equilibrium allocation when there's no trade $C_t = Y_t$ and solve for the interest rate

$$\begin{aligned} 1+r_A &= \frac{1}{\beta}\left(\frac{Y_2}{Y_1}\right)^{\frac{1}{\sigma}} \\ 1+r_A^* &= \frac{1}{\beta}\left(\frac{Y_2^*}{Y_1^*}\right)^{\frac{1}{\sigma}} \\ r_A^* &< r_A \end{aligned}$$

The home country has a higher income growth. Hence, if the Home autarky interest rate was the same as the Foreign, the home country would want to borrow in period 1. But it is not possible to borrow in autarky, hence the Home autarky interest rate must therefore increase to reduce the borrowing motive.

extra How can we be sure that borrowing motive will go down, or equivalently, the saving motive will go up, when the interest rate increase? From above we know that the substitution effect tends to reduce C_1 and thus increase saving, and since the Home country wants to borrow if the interest rate equals the Foreign autarky rate r_A^*, the combined income and wealth effect is negative ($Y_2 > C_2$ as long as you are a borrower), which also reduces C_1.

extra In autarky, income and wealth exactly offset each other, and we are left with only substitution effect. Consequently, in a graph with r along the vertical axis and savings along the horisontal, the slop of the savings schedule will always be positive at the autarky interest rate. Below the autarky interest rate, we have negative savings and the income+wealth effect is negative, hence the slope will be positive. But for r above the autarky rate, the income effect+wealth effect is positive and the slope will turn negative at sufficiently high r (provided $\sigma < 1$). But it will never cross the vertical axis twice, since there is only one autarky interest rate!

extra Note that the autarky interest rate is bounded below at -1

$$r_A = \frac{1}{\beta}\left(\frac{Y_2}{Y_1}\right)^{\frac{1}{\sigma}} - 1$$

since $\frac{1}{\beta}\left(\frac{Y_2}{Y_1}\right)^{\frac{1}{\sigma}} > 0$.

(b) *Suppose the world market consists of these two countries. State the equilibrium condition, and show in a graph how the interest rate will be determined. Which country will run a current account surplus in period 1? Intuitively, what are the gains from trade?*

solution Since there is no capital in this model, the world saving in period 1 S_1^w has to be zero, i.e.

$$S^w = S + S^* = 0$$

where $S = Y_1 - C_1$ and $S^* = Y_1^* - C_1^*$.Note that in a model without capital, the current account equals saving $CA_1 = S_1$. The home and foreign saving depends on the interest rate. The equilibrium interest rate r is the rate that makes

$$S(r) = -S^*(r)$$

Home country will run a current account deficit and Foregin a surplus.Trade allows countries smooth consumption intertemporally by investing in the international financial market. Since the autarky allocation always is possible (you can always choose to consume your income) trade simply increases the consumption possibilities, and the country cannot be worse.

(c) *Using the answer from question* 2. *what happens to welfare in the home country if the foreign country's output growth increases (Y_2^* up)?*

solution If Y_2^* increases r_A^* goes up and r goes up. Since Home is a borrower in period 1 its welfare goes down (worsening of terms of trade).

6. *We now extend the model to include a production side. Each country has access to the same technology and the technology does not change. The production function is* $Y = F(N, K)$ *where* N *and* K *are respectively the inputs of labor and capital. The production function is homogeneous of degree one and has standard neoclassical properties. The labor input is given exogenously and is the same in both countries and both periods. Each country has inherited a capital stock from the past,* K_1 *and* K_1^* *respectively, that can be used in production in period 1. The capital stock can be augmented by investment in period 1, which then adds to the input of capital in period 2,* K_2 *and* K_2^*. *At the end of period 2 the remaining capital stock is consumed. The budget constraint of the home country can then be written*

$$C_1 + I_1 + \frac{C_2 + I_2}{1+r} = Y_1 + \frac{Y_2}{1+r}$$

where $I_1 = K_2 - K_1$ *and* $I_2 = -K_2$. *Explain how the home country's investment demand in period 1 is determined and what the inclusion of capital means for the relationship between the world interest rate and the home country's current account in the first period.*

Solution We assume zero depreciation. Home country's investment demand is derived from the demand for date 2 capital K_2. The optimal capital is pinned down by the equalization of return on domestic real capital and the golbal interest rate:

$$F_K(N, K_2) = r$$

and investment demand is (assuming zero depreciation):

$$I_1 = K_2 - K_1$$

The current account is now given by $CA_1 = S_1 - I_1$

- Effect of r on I_1: If the interest rate increases, investment demand goes down. Marginal productivity of capital is now initialy lower than the world interest rate. The country will therefore save less in domestic capital and more in the international financial market, until equalization of rate of return is restored.
- Effect of r on S_1: We discussed the effect of above. But because of capital there is an additional effect. Remember that we have separation of investment and savings decision. This means that we can think of the agent's problem of making optimal investment, consumption and savings choices in two stages: First the optimal date 2 capital stock K_2 is pinned down by the global interest rate. Second, given the optimal K_2, the agent makes optimal consumption-savings decisions subject to the life-time budget constraint

$$C_1 + \frac{C_2}{1+r} = K_1 + Y_1 + \frac{Y_2 - rK_2}{1+r} \equiv M$$

where the RHS is (as before) the present value of resources, denoted by M. We have income and substitution effects on C_1 as before (effects, holding M constant). Is the wealth effect still negative? i.e. what is the effect of and increase in r on M? The wealth effect via the term $Y_2 - rK_2$ is given by $F_K \frac{dK_2}{d_r} - r\frac{dK_2}{d_r} - K_2 = -K_2$ (again because of the envelope condition), hence the wealth effect is given by

$$\frac{dM}{dr} = -\frac{Y_2 - rK_2}{(1+r)^2} - \frac{K_2}{1+r} = -\frac{(Y_2 + K_2)}{(1+r)^2}$$

- Consequently, the income and substitution effects on C_1, are the same as in the model without investment, and the wealth effect is still negative. In addition investment is decreasing in r.

extra We know that the $CA_1(r)$ schedule is increasing in r as long r is not too high. Why? First, the effect via I_1 is always postive. What about the effect via S_1? The substitution effect on C_1 is negative and the combined income and wealth effect on C_1 is given by

$$\underbrace{\frac{\partial C_1}{\partial M} C_2 \frac{1}{(1+r)^2}}_{\text{income effect}} + \underbrace{\frac{\partial C_1}{\partial M}\frac{\partial M}{\partial r}}_{\text{wealth effect}} = \frac{\partial C_1}{\partial M}\frac{1}{(1+r)^2}(C_2 - Y_2 - K_2)$$

So, in autarky we have that $C_2 - Y_2 - K_2 = 0$, hence, income+wealth effect is zero (as in the model without capital). When we run a current account deficit in period 1, we must have a current accunt surplus in period 2: $CA_1 < 0 \Rightarrow CA_2 = Y_2 - C_2 + K_2 > 0$, and the income+wealth effect on C_1 is negative. Hence, the CA_1 is increasing in r when $CA_1 < 0$. When $CA_1 > 0$ then $CA_2 = Y_2 - C_2 + K_2 < 0$, and the income+wealth effect on C_1 is positive. If r is sufficiently above the autarky interest rate, the slope of $CA_1(r)$ could turn negative, provided $\sigma < 1$.

7. *Compare the capital stocks of the two countries in the second period. Suppose the home country has inherited more capital than the foreign country ($K_1 > K_1^*$). What does this imply for the current accounts in the two periods?*

solution The capital stock in period 2 will be the same in both countries. Suppose that $K_1 = K_1^*$. This means that the two countries are equal, and the only equilibrium is that $CA_1 = CA_1^*$. Suppose, home country gets more capital.

(1) what happens to C_1? An increase in K_1 will increase the RHS of the budget constraint

$$C_1 + \frac{C_2}{1+r} = K_1 + Y_1 + \frac{Y_2 - rK_2}{1+r}$$

because $Y_1 + K_1 = F(K_1) + K_1$ goes up. The consumer is richer and will therefore increase consumption but less than the increase in $Y_1 + K_1$ (want's to smooth)

(2) The current account is given by

$$\begin{aligned} CA_1 &= S_1 - I_1 \\ &= Y_1 - C_1 + K_1 - K_2 \\ &= Y_1 + K_1 - C_1 - K_2 \end{aligned}$$

since K_2 is unchanged and C_1 increases less than the increase in $Y_1 + K_1$ the current account improves. Hence the CA schedule shifts to the right, and we can use the graph to determine the equilibrium. Home (Foreign) country will run a current account surplus (deficit) in period 1 and vice versa in period 2.

(3) Note that this implies that the world market interest rate is higher than the Home autarky interest rate. The autarky interest rate defined as the interest rate that makes investment=savings (must hold in a closed economy)

$$\begin{aligned} CA_1(r_A) &= 0 \\ S_1(r_A) &= I(r_A) \end{aligned}$$

In the graph we find r_A as the intersection between the current account schedule and the vertical axis. Since marginal product of capital is always equal to the interest rate, the period 2 capital stock in the Home country will be higher in autarky than in an open economy.

8. *Finally, suppose international borrowing and lending had not been possible. From the point of view of wage earners in the home country, would this be an advantage or a disadvantage?*

solution marginal productivity of labor (wages) are increasing in the capital stock. Nothing happens to wages in period 1, since the initial capital stock is determined from the past. But period 2 wages in the home country will be lower than under autarky, since some of the savings that in autarky went to capital investment are now invested in the international financial market.

Part 2

In this problem we consider an infinite horizon model with a representative agent and perfect foresight. Each period, the agent must obey the following budget constraint:

$$C_s + B_{s+1} = Y_s + (1 + r)B_s$$

1. Based on the fact that the budget constraint holds for every period from t to $t+T$, show that this implies

$$(1+r)B_t = \sum_{s=t}^{t+T} \left(\frac{1}{1+r}\right)^{s-t} (C_s - Y_s) + \frac{B_{t+T+1}}{(1+r)^T}$$

Solution Since the period-by-period budget constraint holds for every period t to $t+T$ we have

$$\begin{aligned}
B_t &= \frac{1}{1+r}(C_t - Y_t + B_{t+1}) \\
B_{t+1} &= \frac{1}{1+r}(C_{t+1} - Y_{t+1} + B_{t+2}) && (4) \\
B_{t+2} &= \frac{1}{1+r}(C_{t+2} - Y_{t+2} + B_{t+3}) && (5) \\
&\ \ \ \ldots && (6) \\
B_{t+T} &= \frac{1}{1+r}(C_{t+T} - Y_{t+T} + B_{t+T+1}) && (7)
\end{aligned}$$

Using these constraints, start by re-writing the period t budget constraint

$$(1+r)B_t = C_t - Y_t + B_{t+1}$$

Inserting for B_{t+1} using eq 4

$$(1+r)B_t = C_t - Y_t + \underbrace{\frac{1}{1+r}(C_{t+1} - Y_{t+1} + B_{t+2})}_{B_{t+1}}$$

Continue by substitute for B_{t+2} using eq 5

$$\begin{aligned}
(1+r)B_t &= C_t - Y_t + \underbrace{\frac{1}{1+r}\left[C_{t+1} - Y_{t+1} + \underbrace{\frac{1}{1+r}(C_{t+2} - Y_{t+2} + B_{t+3})}_{B_{t+2}}\right]}_{B_{t+1}} \\
&= C_t - Y_t + \frac{C_{t+1} - Y_{t+1}}{1+r} + \frac{C_{t+2} - Y_{t+2}}{(1+r)^2} + \frac{B_{t+3}}{(1+r)^2}
\end{aligned}$$

and continue up to period T.

2. Explain the intuition behind

$$\lim_{T\to\infty} \frac{B_{t+T+1}}{(1+r)^T} = 0$$

solution First: why not $\lim_{T\to\infty} \frac{B_{t+T+1}}{(1+r)^T} < 0$? To rule out Ponzi schemes, in which the agent could pay interest on existing debt by issuing new debt for ever. If this was allowed there would never be any transfers from the agent to the lenders (since the interest payments due each period would be financed by borrowing more), and the present value of consumption is larger than the present value or resources:

$$\sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} C_s > \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)B_t$$

This would require the lenders to hand over resources for free to the agent. The lenders would not accept this scheme. Why not $\lim_{T\to\infty} \frac{B_{t+T+1}}{(1+r)^T} > 0$? Not really a constraint, but it follows from utility maximization, so we simply impose it immediately. If > 0 the consumer does not spend all its resources in present value terms:

$$\sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} C_s < \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)B_t$$

and would hand over resources for free. Intuitively, the agent could increase consumption and not violate the life-time budget constraint. (See Obstfeldt and Rogoff pp. 63-66 for more on this)

3. Impose this restriction and assume that $C_s = cY_s$ and $Y_s = (1+g)^{s-t} Y_t$.

(a) Find the intertemporal budget constraint for this case (when $g < r$).

solution The intertemporal budget constraint (aka life-time budget constraint) is found by letting $T \to \infty$ and imposing the restriction $\lim_{T\to\infty} \frac{B_{t+T+1}}{(1+r)^T} = 0$

$$(1+r)B_t = \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} (C_s - Y_s)$$

if we further assume that $C_s = cY_s$ and $Y_s = (1+g)^s Y_t$ we get

$$\begin{aligned}
(1+r)B_t &= Y_t(c-1) \sum_{s=t}^{\infty} \left(\frac{1+g}{1+r}\right)^{s-t} \\
(1+r)B_t &= Y_t(c-1) \frac{1+r}{r-g} \\
\frac{c-1}{r-g} &= \frac{B_t}{Y_t}
\end{aligned}$$

where we have used that the infinite sum $\sum_{s=t}^{\infty} \left(\frac{1+g}{1+r}\right)^{s-t}$ converges when $g < r$ ($\Rightarrow (1+g)/(1+r) < 1$).

(b) Imagine that keeping consumption at a fixed share c of output indeed is the optimal consumption-choice of a representative agent. Is c above or below one?

solution Use the intertemporal budget constraint to solve for the maximal sustainable level of c.

$$c = (r - g)\frac{B_t}{Y_t} + 1$$

If the agent initially has positive assets $B_t > 0$ then $c > 1$. If the agent initially has debt $B_t < 0$ then $c < 1$

(c) Assume $g = 0$. What does the time-profile of B_t look like for a given value of c?

solution if $g = 0$ then $Y_s = Y_t = Y$ is constant. We get

$$c = r\frac{B_t}{Y} + 1$$

so

$$C = cY = rB_t + Y$$

Hence the agent consumes the interest on the asset each period and the B_t stays constant. The evolution of B_{t+1} is given from the period-by-period budget constraint

$$\begin{aligned} B_{t+1} &= Y - C + (1 + r)B_t \\ &= Y - (rB_t + Y) + (1 + r)B_t \\ &= B_t \end{aligned}$$

and so on. Hence, B_t is constant. This amounts to consuming income Y plus (minus) interest payment on initial asset (debt) rB_t each period.

(d) Assume $g > 0$ (but also $g < r$). What does the time-profile look like now?

solution The solution for c is now

$$c = (r - g)\frac{B_t}{Y_t} + 1$$

B_{t+1} is given by

$$\begin{aligned} B_{t+1} &= Y_t - C_t + (1 + r)B_t \\ &= Y_t(1 - c) + (1 + r)B_t \\ &= (r - g)B_t + (1 + r)B_t \\ &= (1 + g)B_t \end{aligned}$$

B_{t+2} is given by

$$\begin{aligned} B_{t+2} &= Y_{t+1} - C_{t+1} + (1+r)B_{t+1} \\ &= (1+g)Y_t - c(1+g)Y_t + (1+r)(1+g)B_t \\ &= (1+g)(Y_t - C_t + (1+r)B_t) \\ &= (1+g)B_{t+1} \end{aligned}$$

and so on. Hence B_s grows at the same rate as income Y_s,keeping B_s/Y_s constant. Consumption in period s is given by

$$\begin{aligned} C_s &= cY_s = (r-g)\frac{B_t}{Y_t}Y_s + Y_s \\ &= (r-g)B_t(1+g)^{s-t} + Y_s \\ &= (r-g)B_s + Y_s \end{aligned}$$

Hence consumption in period s equals current income plus the growth-adjusted interest rate on current asset.

4. Now assume (as in question 6 of the first problem) that output is a function of the capital stock, $Y_t = A_t F(K_t)$. The utility function is specified as $U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s)$. Use the period s budget constraint to insert for C_s in the utility function.

(a) Find the first-order condition with respect to K_{t+1} and B_{t+1}

solution The period-by-period budget constraint is now

$$C_s + B_{s+1} + K_{s+1} = A_s F(K_s) + K_s + (1+r)B_s$$

substitute this into the objective function to get the maximization problem

$$\max_{\{K_{s+1},B_{s+1}\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} u(A_s F(K_s) + K_s + (1+r)B_s - B_{s+1} - K_{s+1})$$

The foc wrt B_{t+1} is

$$u'(C_t) = \beta(1+r)u'(C_{t+1})$$

and wrt $K_{t+1}$is

$$u'(C_t) = \beta(1 + A_s F'(K_{t+1}))u'(C_{t+1})$$

combining them gives

$$A_s F'(K_{t+1}) = r$$

So we have the standard Euler equation for consumption + the optimal investment condition.

(b) Suppose productivity is constant $A_s = A_t$ for all $s \geq t$ and that, by coincidence, $\beta(1+r) = 1$. Describe the time-profiles of consumption, investment and the current account (you can assume that initial net foreign assets, B_t, are zero).

solution Consumption is constant in all periods and investment and current account is equal to zero in all periods but the first. Since $\beta(1+r) = 1$ consumption is constant. Let K^* be the capital stock that equalizes the marginal product of capital with world market interest rate r

$$AF'(K^*) = r$$

investment in the initial period t is $I_t = K_{t+1} - K_t = K^* - K_t$, the sign depends on the initial level of capital. Investment in all future periods is zero. What about the current account $CA_s = B_{s+1} - B_s$? In the initial period t it is given by

$$CA_t = B_{t+1} = Y_t - C^* - I_t$$

and in all future periods $s > t$ it is given by

$$CA_s = B_{s+1} - B_s = Y^* - C^* + rB_s$$

The only possible value is $CA_s = 0$ and thus $B_s = B_{t+1}$ for all $s \geq t+1$ Why? If CA_s is not zero the country will accumulate debt or assets for ever, which either violates the no-Ponzi scheme condition (in the case of debt) or is suboptimal (in the case of assets), thus violating the condition $\lim_{T\to\infty} \frac{B_{t+T+1}}{(1+r)^T} = 0$. To see this look at the growth rate of B_{s+1} from the current account. If $Y^* - C^* > rB_s$

$$\frac{B_{s+1} - B_s}{B_s} = r + \frac{Y^* - C^*}{B_s}$$

When B_s grows for ever, the term $\frac{Y^*-C^*}{B_s}$ goes to zero, and in the limit B_s grows at rate r. The initial current account CA_t depends on the initial capital stock. If $K_t = K^*$ then also $CA_t = 0$. If the country has $K_t < K^*$, then it will run a initial deficit, and if $K_t > K^*$ it runs a surplus (same argument as in question 7 in problem set 1)

(c) Sketch the effects on consumption, investment and the current account from

i. An unexpected temporary increase in productivity in period $t+1$ (that only lasts one period)

solution Variables will jump in period t+1 (nothing happens in period t since the change is unexpected). When productivity increases only in period $t+1$, investment does not react (since marginal productivity of capital is back at its normal level the next period). The country thus gets a one-time increase in output. Consumption still constant, but jumps to a higher level.

The jump is less than the one-time increase in output because of consumption smoothing. The country saves a fraction of the temporary high output in international financial markets, hence the current account is positive in period $t + 1$ and then zero.

ii. A temporary increase in productivity in $t+1$ (that only lasts one period) that becomes known at the beginning of period t

solution Now investment reacts. The country invests in capital in period t to take advantage of high productivity in the next period. Consumption increases immediately to a permanent higher level. To finance both more consumption and investment, the country borrows from abroad by running a current account deficit in period t. In period $t + 1$ investment is negative since the capital stock returns to K^* in period $t + 2$ and zero thereafter. The current account in period $t + 1$ is positive and zero thereafter.

iii. An unexpected permanent increase in productivity

solution Investment adjust immediately in period t and then returns to to zero. Consumption increases permanently. Output increases immediately due to higher productivity. In period $t+1$ in increases even further since the capital stock is now higher. Consumption therefore increases more than output in period t so savings goes down. Both higher investment and lower savings gives a current account deficit in period t and zero thereafter.