ECON 4330
solution proposal: Seminar 2

1 Short look at OLG

This might not be covered in class, but the model should be known so we do a simple, but instructive, exercise.

1. We use a simple OLG model where consumers live for two periods. They have exogenous labor income only in the first period and save for old age at a world interest rate r. The instantaneous utility function is log utility and the problem of generation t is given by:

$$\begin{aligned} \max_{c_t^y, c_{t+1}^o} \quad & U_t = log(c_t^y) + \beta log(c_{t+1}^o) \\ s.t \quad & c_t^y + b_{t+1} = y_t \\ & c_{t+1}^o = (1+r)b_{t+1} \end{aligned} \tag{1}$$

   (a) Solve for the optimal savings for generation t.
   Solution
   From f.o.c:

$$c_t^y = \frac{c_{t+1}^o}{\beta(1+r)} \tag{2}$$

   Inserted in the first period bc:

$$\begin{aligned} \frac{c_{t+1}^o}{\beta(1+r)} + b_{t+1} &= y_t \\ y_t - b_{t+1} &= \frac{1}{\beta(1+r)}(1+r)b_{t+1} \\ y_t &= (1+\frac{1}{\beta})b_{t+1} \\ b_{t+1} &= \frac{\beta}{1+\beta} y_t \end{aligned} \tag{3}$$

Assume that income grows by a fixed rate, g, every period such that $y_t = (1+g)^t y_0$. Solve for the net savings(savings of the young minus dis-savings of the old) in each period.

(b) How does the time-profile of the net savings of the country depend on the growth rate?
Solution
Net savings $s_t = b_{t+1} - b_t$ since dissavings of the old is equal their savings from last period. Inserting for the savings rule we get:

$$\begin{aligned} s_t &= \frac{\beta}{1+\beta}(y_t - y_{t-1}) \\ s_t &= \frac{\beta}{1+\beta}(1+g-1)y_{t-1} \\ s_t &= \frac{\beta g}{1+\beta}y_{t-1} \end{aligned} \tag{4}$$

(c) How does this compare to the infinite horizon case from the last seminar?
Solution
In the last exercise high growth ment low savings because you wanted to borrow money to smooth consumption. Here each agent can not lend and pay back far into the future because they die. People only save for old age and since each generation is richer than the last aggregate savings will increase with time. Countries with high growth will save more.

2 Transfer problem(Revised version of Exam 2015)

Consider a model with only tradable goods, but differentiated by country. Consumption c in the home country is the aggregate of home produced good c_h and foreign good c_f,

$$c = \xi c_h^{\omega} c_f^{1-\omega}, \tag{5}$$

where $1 \geq \omega \geq 0.5$ and $\xi = \omega^{-\omega}(1-\omega)^{-(1-\omega)}$. Households derive utility $u(c)$ from consuming c. The home country produces y_h of the home good at price p_h. The price of the foreign good is p_f, where the foreign good is the numeraire, so that $p_f = 1$. The budget constraint is then equal to

$$p_h y_h = p_h c_h + p_f c_f. \tag{6}$$

1. Write down the household's optimization problem and solve for the demmand of the two goods.
   Solution:

$$\begin{aligned} \max_{\{c_h, c_f\}} \quad & u(c) \\ s.t. \quad & p_h y_h = p_h c_h + p_f c_f \end{aligned}$$

   Setting it up as a lagrangean and using $p_f = 1$ gives us:

$$L = u(c) - \lambda(p_h c_h + c_f - p_h y_h)$$

   and the first order conditions:

$$\begin{aligned} \frac{\partial L}{\partial c_h} &= u'(c)\xi\omega c_h^{\omega-1} c_f^{1-\omega} - \lambda p_h &= 0 \\ \frac{\partial L}{\partial c_f} &= u'(c)\xi(1-\omega) c_h^{\omega} c_f^{-\omega} - \lambda &= 0 \end{aligned}$$

   solving for consumption of the two goods we get:

$$\begin{aligned} c_h &= \omega y_h \\ c_f &= (1-\omega) p_h y_h \end{aligned}$$

2. Show that the share of home goods in total consumption is:

$$\psi = \frac{\omega}{p_h^{1-\omega}} \tag{7}$$

   such that

$$c_h = \psi c, \tag{8}$$

   Solution:
   Start out with counsumption of the home good and divide by aggregate consumption.

$$\begin{aligned} \psi &= \frac{c_h}{c} \\ \psi &= \frac{c_h}{\xi c_h^{\omega} c_f^{1-\omega}} \\ \psi &= \xi c_h^{1-\omega} c_f^{\omega-1} \\ \psi &= \omega^{\omega}(1-\omega)^{(1-\omega)}(\omega y_h)^{1-\omega}((1-\omega)p_h y_h)^{(\omega-1)} \\ \psi &= \omega^{\omega+1-\omega}(1-\omega)^{(1-\omega+\omega-1)} y_h^{1-\omega+\omega-1} p_h^{\omega-1} \\ \psi &= \omega p_h^{\omega-1} \\ \psi &= \frac{\omega}{p_h^{1-\omega}} \end{aligned}$$

3. Consider now the case of two symmetric countries, that is the foreign country has the preferences

$$c^* = \xi (c_f^*)^{\omega} (c_h^*)^{1-\omega} \tag{9}$$

and produces y_f goods. Write down the foreign household's optimization problem and solve for foreign consumption of the two goods.
Solution:

$$\max_{\{c_h^*, c_f^*\}} \quad u(c^*)$$

$$s.t. \qquad y_f = p_h c_h^* + c_f^*$$

Setting it up as a lagrangean and using $p_f = 1$ *gives us:*

$$L = u(c^*) - \lambda (p_h c_h^* + c_f^* - y_f)$$

and the first order conditions:

$$\begin{aligned} \frac{\partial L}{\partial c_f^*} &= u'(c^*)\xi\omega (c_f^*)^{\omega-1} (c_h^*)^{1-\omega} - \lambda &= 0 \\ \frac{\partial L}{\partial c_h^*} &= u'(c^*)\xi(1-\omega)(c_f^*)^{\omega}(c_h^*)^{-\omega} - \lambda p_h &= 0 \end{aligned}$$

solving for consumption of the two goods we get:

$$\begin{aligned} c_f^* &= \omega y_f \\ c_h^* &= (1-\omega)\frac{y_f}{p_h} \end{aligned}$$

4. Find the share of home goods (produced in the home country) in total foreign consumption, $(1-\psi^*)$, such that:

$$c_h^* = \psi^* c^*, \tag{10}$$

Solution:

Same procedure as above:

$$\psi^* = \frac{c_h^*}{c^*}$$
$$\psi^* = \frac{c_h^*}{\xi(c_f^*)^\omega (c_h^*)^{1-\omega}}$$
$$\psi^* = \xi^{-1}(c_h^*)^\omega (c_f^*)^{-\omega}$$
$$\psi^* = \omega^\omega(1-\omega)^{(1-\omega)}((1-\omega)\frac{y_f}{p_h})^\omega((\omega y_f)^{-\omega}$$
$$\psi^* = \omega^{\omega-\omega}(1-\omega)^{(1-\omega+\omega)} y_f^{\omega-\omega} p_h^{-\omega}$$
$$\psi^* = (1-\omega)p_h^{-\omega}$$
$$\psi^* = \frac{(1-\omega)}{p_h^\omega}$$

5. Solve for the price p_h. Hint: Use the goods market clearing condition $c_h + c_h^* = y_h$.
   Solution:
   Start out with the equilibrium condition, insert the solutions you found for c_h and c_h^* and solve for p_h:

$$c_h^* + c_h = y_h$$
$$(1-\omega)\frac{y_f}{p_h} = (1-\omega)y_h$$
$$p_h = \frac{y_f}{y_h}$$

6. Now suppose that the home country receives a transfer T from the foreign country, so that it can spend now $p_h y_h + T$ and the foreign country can spend $p_f y_f - T$. Solve for the price p_h. Hint: Use again the goods market clearing condition $c_h + c_h^* = y_h$. The right hand side is still the same, we don't produce more.

   Solution:
   The solutions we found earlier for c_h and c_h^* are still valid only $y_f - T$

replaces y_f and $y_h + \frac{T}{p_h}$ replaces y_h:

$$\begin{aligned} c_h^* + c_h &= y_h \\ (1-\omega)\frac{y_f - T}{p_h} + \omega(y_h + \frac{T}{p_h}) &= y_h \\ (1-\omega)\frac{y_f - T}{p_h} + \omega\frac{T}{p_h} &= (1-\omega)y_h \\ p_h &= \frac{y_f}{y_h} + \frac{(2\omega - 1)T}{y_h(1-\omega)} \end{aligned}$$

7. How does the price p_h depend on T?
Solution:
$\omega > 0.5$ so the price increases when T is positive. Formally we can find:

$$\frac{\partial p_h}{\partial T} = \frac{(2\omega - 1)}{y_h(1-\omega)} > 0$$

8. Is there a value of ω where p_h is independent from T? Why?
Solution:
If $\omega = 0.5$ the price would be independent of T. That is because $\omega = 0.5$ means that there is no home bias in goods. Both countries like both goods equally. When ω is larger than 0.5 it means that home consumers prefer the home good, when they are awarded a transfer demand for home goods increase because the fall in demmand from foreign consumers is smaller than the increase in demand from home consumers. The opposite would be true if we had a foreign bias ($\omega < 0.5$).

9. What happens with ψ and ψ^*?
Both ψ and ψ^* depend negatively on p_h, so they would both fall. Home goods become relatively more expensive for both consumers, so their share of home goods in aggregate consumption fall.

10. What happens to total consumption of the good produced in the home country?
Solution:
It remains unchanged. Production is fixed at y_h, so the total amount consumed must still be $c_h + c_h^* = y_h$

11. Is the change in the price p_h a burden or a benefit for the foreign country?
Solution:

The foreign country takes a direct loss because of the transfer T, but there will be a second effect through the price change. We can analyze that by taking the derivative of their total utility with respect to the price of the home good:

$$\frac{\partial u(c^*)}{\partial p_h} = u'(c^*)\frac{\partial c^*}{\partial p_h}$$
$$\frac{\partial u(c^*)}{\partial p_h} = u'(c^*)\epsilon(\omega(y_f - T))^{\omega}((1-\omega)\frac{1}{p_h}(y_f - T))^{1-\omega}(1-\omega)(y_f - T)(\frac{-1}{p_h^2}) < 0$$

smaller than zero as long as T is smaller than y_f which is reasonable to assume as it would lead to negative consumption. There is a negative price effect because one of the goods the foreign consumer is buying is becoming more expensive.