ECON 4335 Economics of Banking, Fall 2015

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Supplement to the lecture on the Diamond-Dybvig model

The main results of this model are provided by Tirole in "The Theory of Corporate Finance"; chapt. 12. However, there are still some issues that might be a bit confusing. There is no need going through the model here but some main results are briefly outlined.

Each agent is endowed with one unit of "wealth" that can be consumed or invested. No consumption takes place at $t = 0$. Ex ante (at $t = 0$) no agent (among a continuum of agents) knows what type she will be at $t = 1$; either an early consumer (with probability λ) or a late consumer (with probability $1 - \lambda$). The probability distribution is common knowledge. The types of the agents are independent and identically distributed. This is equivalent to say that type realizations are idiosyncratic (diverifiable, no systematic (or "macro") risk). Types cannot be verified by a third party and therefore no insurance company will ex ante offer insurance contracts based on such information.

Being an early consumer is considered as a liquidity shock: the agent is exposed to an early liquidity need or demand for cash. The objective of any agent in period t=0 is to maximize expected ex ante utility. There is no time discounting as it is not necessary for the results.

$$U = \lambda u(C_1) + (1 - \lambda)u(C_2)$$

where $u(C)$ is an ordinary VNM (Von Neumann Morgenstern) utility function, with C as consumption per capita: $u(C)$ is increasing, strictly concave (so agents are risk averse) with $u'(0) = -\infty$, and a coefficient of relative risk aversion (or an intertemporal elasticity of substitution) $\left|\frac{Cu''(C)}{u'(C)}\right| > 1$ for all C. This assumption ensures that an agent prefers a compressed consumption profile.

Ex ante there are two investment opportunities: one short-term project, which is a pure storage technology – each unit that an agent invests in this project will leave the agent with one unit

[1]This note is almost entirely based on the note written by Jon Vislie for the Fall 2014 Banking course.

after one period, or a long-term project with a gross return reaped after two periods; the gross return is $R > 1$. This long-term project can, if necessary, be stopped, interrupted or liquidated at $t = 1$, with a gross return $l < 1$ per unit invested. Hence there is a real loss caused by liquidation, as a fraction $1 - l$ will disappear. The trade-off is therefore between early need for cash and the benefit from investing in the long-term project. Let $I \in [0, 1]$ be the fraction of one's wealth that is invested in the long-term project.

SOCIALLY OPTIMAL ALLOCATION

A sociall optimal allocation can be characterized as follows. At the ex-ante stage, a planner will solve the following risk-sharing/investment program, while relying on the law of large numbers – i.e. the planner knows the fraction of early consumers, but not exacly which person will be an early or a late consumer. As agents are ex ante identical, the planner may solve the problem of a representative agent:

$$\max \{\lambda u(C_1) + (1-\lambda)u(C_2)\} \quad s.t. \quad \begin{cases} \lambda C_1 & = 1 - I \\ (1-\lambda)C_2 & = RI \end{cases}.$$

The expected consumption in the first period is λC_1 and is made up of short-term investment $1 - I$, whereas expected consumption in period 2 is $(1 - \lambda)C_2 = RI$, being equal to the gross return from the long-term investment. Combining these to constraints:

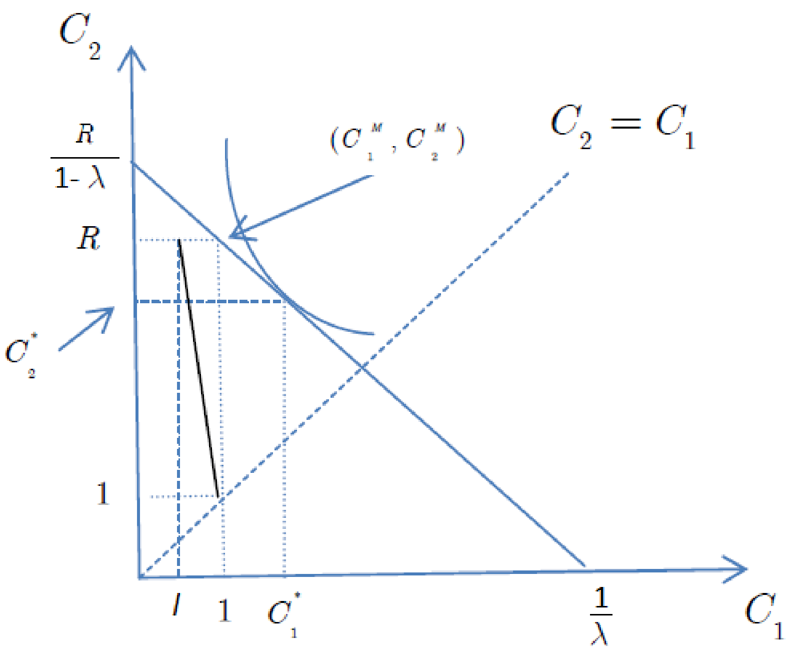
$$\lambda C_1 = 1 - \frac{1-\lambda}{R} C_2 \leftrightarrow \lambda R C_1 + (1-\lambda)C_2 = R \rightarrow \frac{dC_2}{dC_1} = \frac{\lambda R}{1-\lambda}.$$

This is an efficiency-locus or the frontier of the opportunity set. Hence the first-best solution is found by maximizing expected utility subject to this constraint.

It is a straightforward optimization program that can easily be illustrated as follows: in the optimal solution the slope of an indifference curve is equal to the slope of the efficiency locus; i.e., we must have:

$$\frac{\lambda u'(C_1^*)}{(1-\lambda)u'(C_2^*)} = \frac{\lambda R}{1-\lambda} \leftrightarrow \frac{u'(C_1^*)}{u'(C_2^*)} = R.$$

where the last equality says that the required rate of return from saving should be equal to the net return from long-term investment. Hence we have: $u'(C_1^*) = Ru'(C_2^*)$.



The assumption $\left|\frac{Cu''(C)}{u'(C)}\right| > 1$ is equivalent to $\frac{d}{dC}(Cu'(C)) < 0$. Therefore $R > 1$, implies $u'(1) > Ru'(R)$.

$(C_1, C_2) = (1, R)$ is a feasible allocation and lies on the efficiency locus – hence $\frac{u'(1)}{u'(R)} > R$ implies that $R > C_{:2}^*$ and $C_1^* > 1$. As $R > 1$, we have $u'(C_1^*) > u'(C_2^*)$, and as and $u'' < 0$, $u'(C_1^*) > u'(C_2^*)$ implies $C_1^* < C_2^*$. This will also make the solution so called *incentive efficient* in the sense that no late agent will be motivated to act as an early consumer. A late consumer can always pretend to be impatient; "take out" C_1^* at $t = 1$, and invest this amount in a short-term project with a gross return C_1^* to be consumed at $t = 2$. However, this behavior will incur a utility loss as $C_1^* < C_2^*$.

AUTARKY

Under **autarky** or self-insurance each agent must provide for her own needs by hoarding liquidity to meet a liquidity shock. If the agent becomes an early consumer, liquidity is provided by liquidating the long-term investment along with consuming the return from the short-term project. On the other hand, if she faces no liquidity shock, the short-run investment is rolled over another period, while also reaping the return from the long-term project. In both cases, not knowing one's type will be costly ex post.

Once an agent selects the amount of the long term investment, if she ends up having an early liquidity need, she will have a consumption capacity $C_1 = 1 - I + \lambda I \leq 1$; i.e. the sum of the short-run (liquid) investment and the liquidated value of the long-term investment. If she is not exposed to a shock, then her consumption capacity at $t = 2$, is $C_2 = 1 - I + RI \geq 1$; the short-term investment rolled over (without any net return) and the full return from the illiquid long-term project.

Note that C_1 is equal to one only if $I = 0$, in which case $C_2 = 1$. On the opposite end, if $I = 1$ then $C_1 = l < 1$ and $C_2 = R$. The bold line in the diagram above, between $(1, 1)$ and (l, R), shows all the combinations that an agent can ensure under autarky. Autarky goes with a social loss, as all the combinations ensure a utility lower than C_1^*, C_2^*; hence we have $\lambda C_1 + (1-\lambda)\frac{C_2}{R} < 1$ because in (l, R) we have $\lambda l + (1-\lambda) < 1$ and in $(1, 1)$ we have $\lambda + \frac{1-\lambda}{R} < 1$.

The optimal self-insurance combination C_1^A, C_2^A solves:

$$\max_{I \in [0,1]} \lambda u(C_1) + (1-\lambda) u(C_2) \text{ s.t. } \begin{cases} C_1 = & 1 - (1-l)I \\ C_2 = & 1 + (R-1)I \end{cases}$$

Substituting in the constraints, the function to be maximized becomes:

$v(I) \equiv \lambda u(1 - (1-l)I) + (1-\lambda) u(1 + (R-1)I)$

therefore the first order condition is $v'(I) = -\lambda u'(C_1^A)(1-l) + (1-\lambda)u'(C_2^A)(R-1) = 0$.

$v'(0) = -\lambda u'(1)(1-l) + (1-\lambda)u'(1)(R-1) = u'(1)\left[(1-\lambda)(R-1) - \lambda(1-l)\right]$.

If $\frac{R-1}{1-l} \leq \frac{\lambda}{1-\lambda}$, then $v'(0) \leq 0$ and therefore $I^A = 0$. $\frac{R-1}{1-l} \leq \frac{\lambda}{1-\lambda}$ is the case if R is small and l small. In that case; the agents invest only in the short-term project. On the other hand, if $v'(1) \geq 0$, then $I^A = 1$.

An interior solution $I^A \in (0, 1)$ is characterized by: $\frac{\lambda u'(C_1^A)}{(1-\lambda)u'(C_2^A)} = \frac{R-1}{1-l}$.

Note that ex post, the investment decision is always inefficient. At $t = 1$, $I = 0$ is the optimal investment choice by an impatient agent, whereas $I = 1$ is the best one for a patient one.

FINANCIAL MARKET

A different institutional arrangement is possible: announce at $t = 0$, that a financial market will open at $t = 1$, once the agents have learnt their own type. The financial market allows exchange between agents of different type, say by trading bonds. Will the first-best be achieved? A bond market is opened at $t = 1$. Suppose that having one unit of a bond will entitle you to one unit of consumption at $t = 2$. Then one can go around the direct liquidation of long-term investment if it turns out that an agent is exposed to a liquidity shock, because we now will have no loss due to liquidation. An agent being exposed to a liquidity shock at $t = 1$, will want to **borrow** now so as to be able to consume early, by **selling bonds** in a number equal to the full return from her long-term project. (Note that selling bonds is the mirror image of demanding or buying goods.) This means that an agent with early consumption needs can convert future returns into current consumption by **selling bonds or borrowing**, while the **buyers of these bonds** (the lenders that supply goods at $t = 1$, because they have have a surplus at $t = 1$) are entitled to the return from the long-term investment made by early consumers by giving away or lending the return from their short-term investment. The late agents will have a surplus at $t = 1$, equal the return from the short-term investment, which can be consumed by those having early consumption needs.

Let us look at the financial market equilbrium. As said above; one unit of bond bought at $t = 1$ from an early (or type-1) agent entitles the owner to one unit of late consumption. Now there is no loss from liquidation; we will end up on the efficiency locus, but the risk-sharing will not be optimal because ex ante there cannot be a full set of contingent (Arrow-Debreu) markets. Let p be the price of a bond, which measures the number of period 1 consumption units per unit consumption in period 2. Hence p is the relative price of late consumption (in units of early consumption), whereas the inverse, $\frac{1}{p}$, measures the number of period 2 consumption units per unit consumption in $t = 1$, which is like a rate of gross return from having one unit of a bond. An agent being exposed to a liquidity shock (having early consumption needs) will like to borrow, that is sell RI units of bonds. The amount RI is equal to her future income, that is, the return from her long-term investment that will be reaped at $t = 2$. Each bond is sold at a price p, converting the future income to period 1-consumption, so that the buyer (a late agent) will get this return later. An early agent will then have the following consumption opportunity: while reaping the return from her short-term investment, she gets p units of consumption in period $t = 1$ from the late agents for each unit of bond sold. Hence an early consumer will have: $C_1 = 1 - I + pRI$.

A late agent buys bonds and sells away her short-term return or surplus of goods at $t = 1$, by transforming this surplus to a consumption stream at $t = 2$. The opportunity for a late agent is therefore: $C_2 = RI + \frac{1}{P}[1 - I]$. A late agent will have her own return at $t = 2$, but by selling away, through buying bonds at $t = 1$, the surplus at $t = 1$, which is $1 - I$ units of early consumption, will be converted to $\frac{1-I}{p}$ units of period-2 consumption in the financial market – by trading bonds. We observe that $C_1 = pC_2$.

Suppose that $p > \frac{1}{R}$: then the rate of return from buying a bond is below the rate of return from investing in the illiquid long-term project. A rational investor will then want to choose $I = 1$ because the rate of return from this type of investment is higher than the alternative. That means that at $t = 1$, there is an excess demand for goods or excess supply of bonds. This cannot constitute a market equilibrium. If on the other hand, $p < \frac{1}{R}$, the return from having a bond exceeds the return from the long-term project. Every investor will choose $I = 0$, creating an excess supply of goods or excess demand for bonds at $t = 1$. Not an equilibrium. The return from having a bond exceeds the return from investing in the illiquid project. Therefore in equilibrium we must have $p = \frac{1}{R}$. Note that the market clearing condition in the bond market is: $\pi RI = (1 - \pi)\frac{1-I}{p}$. Tota1 number of bonds supplied is equal to total demand for bonds, at $t = 1$.

From the perspective of one agent, the individual ex ante choice within this arrangement is given by:

$\max_{I\in[0,1]} \lambda u(1 - I + pRI) + (1 - \lambda)u(\frac{1}{p}[1 - I + pRI])$,

Every agent will choose the M(arket)-allocation that satisfies

$$\lambda u'(C_1^M)(pR-1)+(1-\lambda)u'(C_2^M)\frac{1}{p}(pR-1)\begin{cases}>0 & \text{if } pR-1>0\rightarrow I=1\\ =0 & \text{if } pR-1=0\rightarrow I\in[0,1]\\ <0 & \text{if } pR-1<0\rightarrow I=0\end{cases}$$

Because we have $p=\frac{1}{R}$, we get: $\lambda I=(1-\lambda)(1-I)\leftrightarrow I^M=1-\lambda$.

We then have: $C_1^M=1$ and $C_2^M=R$ and $I^M=1-\lambda<1$.

Hence in this market equilibrium we have (as indicated in the previous diagram):

$C_2^M=R>C_2^*>C_1^*>C_1^M=1$ and $I^M>I^*$.

It should not come as a surprise that the market equilbrium Pareto-dominates the autarky equilibrium because in the market equilibrium there is no liquidation:

$C_1^M=1\geq C_1^A$ and $C_2^M=R\geq C_2^A$.

However, the market equilibrium is not ex ante Pareto-efficient because we have not a complete set of risk or insurance markets ex ante. As a result, liquidity risk cannot be properly allocated. The market equilibrium shows higher volatility in consumption than the first best allocation, and because risk is not properly shared, people will take more cautious actions by investing more in the long-term project than what is optimal.

BANKS

The last institutional arrangement is banks. Banks let agents deposit their endowment. The bank will undertake the required asset transformation by investing (lending) in the long-term project, while offering demand deposits to all agents. Each agent is allowed to withdraw C_1^* units if she needs early liquidity (this is equivalent to be promised a short-run rate of interest), or allowed to withdraw C_2^* at $t=2$ (or being offered a higher, long-run rate of interest).

In a **competitive banking equilibrium**, banks compete for deposits. If one bank should offer a deposit contract that differs from the efficient one, another bank will overturn this offer by offering one that is closer to the one that maximizes expected utility. With free entry in the banking industry, bank profits will be zero and equilibrium is characterized by each bank offering the efficient contract. Hence, introducing banks will implement the efficient allocation!

Banks can bridge the gap or eliminate the mismatch between the maturity structure and the population's need for liquidity.

A free-entry banking equilibrium is characterized by banks offering a contract or consumption profile to agents so that:

$\max_{(C_1,C_2)} \lambda u(C_1) + (1-\lambda)u(C_2)$

s.t. $(1-\lambda)C_2 = (1-\lambda C_1)R$

If the bank offers a profile different from the one that maximizes the expected utility, another bank will gain by offering the one that is preferred by depositors. The zero-profit condition says that what is left for late depositors is equal to the long-term return from investment (lending), after having put aside, as liquid reserves, an amount equal to what early consumers are expected to withdraw; hence an amount $1-\pi C_1$ is being lent out at $t=0$, with a gross return equal to R. Note that the zero-profit condition is identical to our efficiency locus. Hence, the competitive equilibrium deposit contract will coincide with the efficient consumption profile.

When agents are exposed to idiosyncratic (diversifiable) liquidity shocks, the allocation of the financial market equilibrium can be improved upon by having banks offering deposits contracts.

Banks or financial intermediaries act like insurance companies and eliminate the cost of maturity mismatch by undertaking the optimal long-term investment. The balance sheet will then look like:

Assets	Liabilities/Debt
Liquid Reserves: λC_1^*	Deposits: 1
Loans: $I = 1-\lambda C_1^*$	

In this equilbrium there is no need for equity: all risk can be fully diversified and there is no credit risk; the borrowers will repay with probability one.

But, and this is the story with bank runs, the desirable properties of the bank solution are highly dependent on the fact that only a fraction λ of depositors withdraw early. The efficient deposit contract constitutes a Nash equilibrium. If any late depositor expects none of the remaining late depositors to withdraw early, no late depositor will have any incentive to deviate; i.e. to withdraw early. On the other hand, as shown by Tirole, there is also another Nash equilbrium where all depositors withdraw early under a system “first-come-first-serve”. In that case, the promised original (efficient contract) cannot be honoured.