ECON 4335 Economics of Banking, Fall 2015

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The derivation of the main result in Tobin's model

The liability side: Deposits (D) + Equity (E)

E is assumed exogenous, while D is a random short-term deposit. Uncertainty can be rationalized by banks not rolling over their loans to our bank, and/or depositors taking the money out randomly – you never know when a depositor needs money. D + E is the amount of money that can be provided as loans. We rule out any interest on deposits – this is just a simplifying assumption.

The asset side: Loans (L) + Required Reserves (kD) + Defensive position (R)

It is assumed that the fraction of deposits kept as required reserves, k, is a fixed constant, and is a government instrument. Loans or the bank's lending policy is a choice variable chosen so as to maximize expected profits, along with defensive position or free reserves.

The loans are less liquid and of longer maturity than deposits; so the bank might get into liquidity problems as a result of unanticipated withdrawals by depositors or by other banks. Reserves beyond those required, called defensive positions, are kept for precautionary reasons, and are highly liquid (cash, deposits at the Central Bank, short-term loans to other banks or government bonds). These positions can be negative if the bank has to borrow in the interbank market or at the Central Bank; normally at a higher funding rate than what the bank can achieve in the market as a lender, with a positive defensive position.

Fixed Deposits. Before turning to the main problem, random supply of deposits, let us, as a benchmark, show the behavior of the bank, with a fixed volume of deposits equal to D_0.

The expected revenue from lending is given, for simplicity, by the strictly increasing and strictly concave revenue function $P(L)$, with $P(0) = 0$, $P(L) > L$, $P'(L) > 0$, $P''(L) < 0$; we also assume $P'(0) = \infty$ and $P'(\infty) = 0$ (to guarantee the that optimal level of loans is neither 0 nor ∞, which is equivalent to guarantee an interior solution). This function can capture the

[1]This note is almost entirely based on the note written by Jon Vislie for the Fall 2014 Banking course.

product of the revenue if the borrowers do not default and the probability that the borrowers do not default.

Profits per year: $\Pi = P(L) + Y(R)$, where we have to take into account the balance sheet condition:

$$D_0 + E = kD_0 + R + L \leftrightarrow R = (1-k)D_0 + E - L$$

The funding cost of being in a negative defensive position is higher than the interest on revenue from a positive defensive position. We ignore fixed costs. We have:

$$Y(R) = \begin{cases} rR & \text{for } R \geq 0 \\ (r+b)R & \text{for } R < 0 \text{ and } b > 0 \end{cases}$$

Hence we can write the profit function as consisting of two parts:

If $R \geq 0$: $\Pi_+(L) = P(L) + r\,[(1-k)\,D_0 + E - L]$

If $R < 0$: $\Pi_-(L) = P(L) + (r+b)\,[(1-k)\,D_0 + E - L]$

The opportunity cost of making loans is r if $R \geq 0$, and $r+b$ if $R < 0$ and $b > 0$. Define a critical volume of loans corresponding to defensive positions exactly equal to zero; $L_c := (1-k)D_0 + E$. At this loan volume, the derivative of the opportunity cost curve is discontinuous. We can illustrate this in Figure 0.1.

Depending on whether $P'(L_c)$ is below r or in between r and $r+b$ or above $r+b$, the optimal amount of loans, L^*, is determined from the optimality condition (= first-order condition):

(1) If $P'(L_c) \leq r$, then $P'(L^*) = r$ for some $L^* \leq L_c$, and $R^* \geq 0$, because $P(L)$ is concave.

(2) If $P'(L_c) \in (r, r+b)$, then $L^* = L_C$, and $R^* = 0$.

(3) If $P'(L_c) > r+b$, then $L^* \geq L_C$, and $R^* \leq 0$.

These alternatives are illustrated in Figure 0.2.

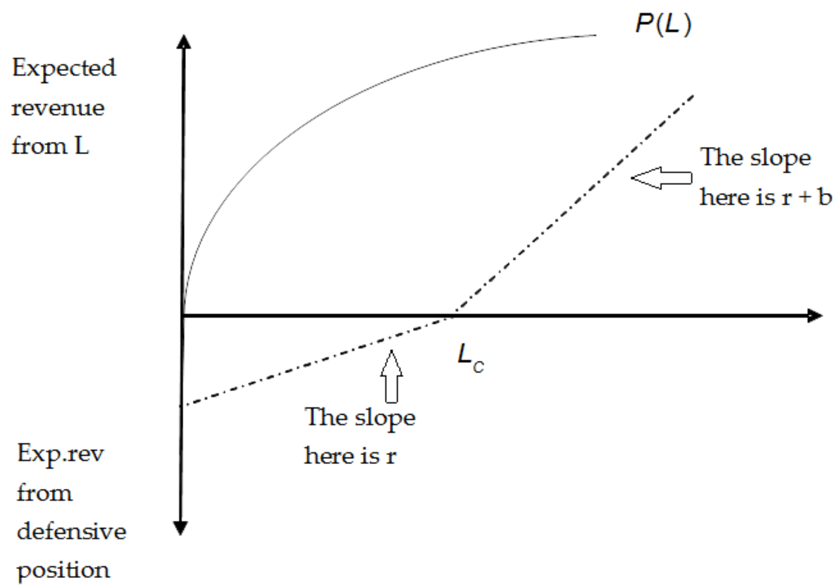


FIGURE 0.1. Revenue and cost of loans.

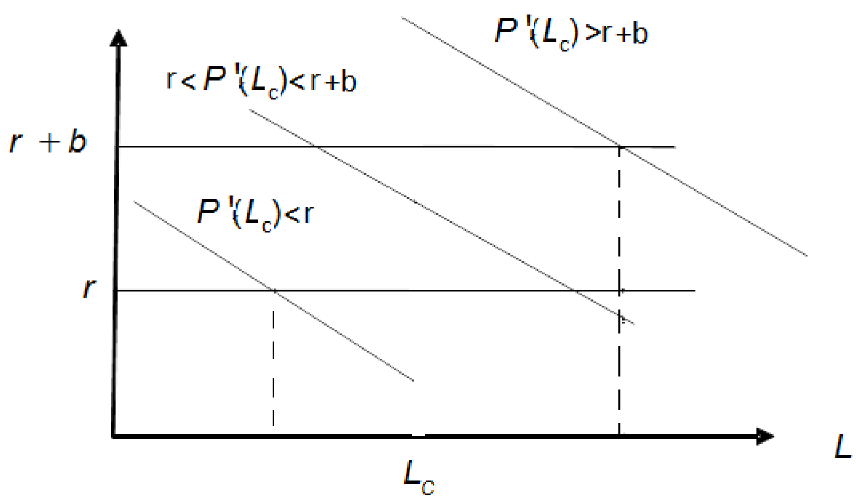


FIGURE 0.2. Marginal revenue(s) of loans.

Random supply of deposits. Let us turn to the most interesting case: random supply of deposits. In this case, the bank has to take into account that withdrawals can be so high that even if the bank extends a small amount of loans, the defensive position might be negative with some probability. As a result, we expect the bank to hold excessive reserves as a buffer against unexpected withdrawals from depositors.

In the diagrams above, for $L < L_C$ the probability of a negative position was zero. When deposit withdrawals are uncertain, the probability of ending up in a negative position might be positive even for a cautious or conservative lending policy. We will show that when we introduce a probability distribution for withdrawals, the discontinuous marginal opportunity cost curve in the certainty case is replaced by a continuous one, showing the expected opportunity cost of making loans. This expected opportunity cost will include an allowance for having a negative defensive position with some probability; this allowance will be higher the higher is L.

The bank is risk neutral, maximizing expected net revenue. Let deposits be random, given by the stochastic variable $D = D_0(1 + X)$, where $X \geq -1$ and $E(X) = 0$; hence $E(D) = D_0$ is a constant. The cumulative probability distribution for X is $F(x) = Pr(X \leq x)$; differentiable and strictly increasing; hence the distribution has a positive density $f(x) = \frac{dF(x)}{dx}$.

L is set before deposits are known; hence defensive position $R = (1 - k)D_0(1 + X) + E - L$ is also a random variable.

Define a critical level of X, denoted y, so that $R \leq 0$ whenever $X < y$.[2]

$R \leq 0$ whenever $XD_0 \leq yD_0 = \frac{L-E}{1-k} - D_0 \rightarrow y = \frac{L-E}{(1-k)D_0} - 1$. (I)

Expected reserves are:

$E(R) = (1 - k)D_0 + E - L = (1 - k)\left[D_0 - \frac{L-E}{1-k}\right] = -yD_0(1 - k)$.

Hence if $X \leq y$, then $R \leq 0$ and the opportunity cost is $r + b$; if instead $X > y$, then $R > 0$ and the opportunity cost equals r.

Therefore $Pr(R \leq 0) = Pr(X \leq y) = F(y) = F\left(\frac{L-E}{(1-k)D_0} - 1\right)$.

Define expected profits per year as:

$\Pi(L, y) \equiv \int_{-1}^{y} \{P(L) + (r + b)[(1 + x)D_0(1 - k) + E - L]\} f(x)dx+$

$+\int_{y}^{\infty} \{P(L) + r[(1 + x)D_0(1 - k) + E - L]\} f(x)dx$

$\Pi(L, y)$ can be rewritten in a more manageable form. Proceeding step by step:

$\Pi(L, y) = \int_{-1}^{\infty} \{P(L) + r[(1 + x)D_0(1 - k) + E - L]\} f(x)dx+$

[2]We use a somewhat different notation than what Tobin does.

$+b\int_{-1}^{y}\{[(1+x)D_0(1-k)+E-L]\}f(x)dx$

$= P(L)+r[D_0(1-k)+E-L]\int_{-1}^{\infty}f(x)dx + rD_0(1-k)\int_{-1}^{\infty}xf(x)dx+$

$+b\int_{-1}^{y}\{(1+x)D_0(1-k)+E-L\}f(x)dx$

using $\int_{-1}^{\infty}f(x)dx = 1$, and $\int_{-1}^{\infty}xf(x)dx = E(X) = 0$:

$= P(L)+r[D_0(1-k)+E-L]+b(1-k)D_0\int_{-1}^{y}\left\{x+1-\frac{L-E}{(1-k)D_0}\right\}f(x)dx$

using (I):

$= P(L)-rD_0(1-k)y+b(1-k)D_0\int_{-1}^{y}\{x-y\}f(x)dx$

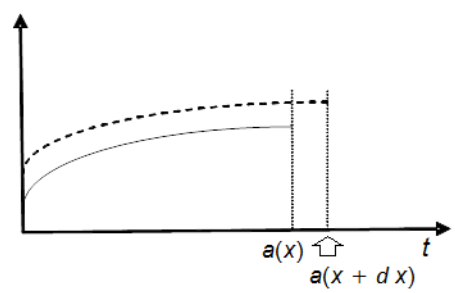
$= P(L)-rD_0(1-k)y+b(1-k)D_0\int_{-1}^{y}xf(x)dx - b(1-k)D_0y\int_{-1}^{y}f(x)dx$

$= P(L)-rD_0(1-k)y+b(1-k)D_0\int_{-1}^{y}xf(x)dx - b(1-k)D_0yF(y)$

To find the level of loans that maximize the expected profits, we need to take into account that y itself is a function of L. As y enters as the upper limit in the integral, we need to take a technical detour.

Technical detour:

Let $G(X) = \int_0^{a(x)} g(t,x)dt$, which can be illustrated as the area between the graph for the g-function and the horizontal axis between the origin and $a(x)$ as below:



Suppose now (this is a bit more general than the problem we are solving) that if x increases, both $a(x)$ and $g(t,x)$ will increase; the upper limit is pushed towards the right along the horizontal axis, from $a(x)$ to $a(x+dx)$ whereas the graph itself will shift upwards for any t; hence we assume (given differentiability) that $\frac{\partial g(t,x)}{\partial x} \equiv g_x(t,x) > 0$ (in our problem we have $g_x(t,x) = 0$). The new curve is the one shown by the dashed line. For a finite increase in x, the increase in the function G is the area between the two graphs for a fixed end-point, plus the area as shown by the rectangle between the old and the new end-point. This is the "explanation" for the Leibniz formula as given by

$$G'(x) = g(a(x), x)a'(x) + \int_0^{a(x)} g_x(t, x)dt,$$

where the first term is the rectangle in the graph, whereas the last term is the area between the graphs for a given end-point as given by the original one. In our problem, the last term is zero; hence we have $G'(x) = g(a(x), x)a'(x)$, which is a rectangle with height equal to the value of g at $a(x)$; i.e. $g(a(x), x)$, multiplied by the increase in the distance between $a(x)$ and $a(x + dx)$ along the horizontal axis as $dx \to 0$. Hence the derivative $G'(x)$ is the increase as shown by a very "thin" rectangle with height $g(a(x), x)$ and very small width $a'(x)dx$; as x increases by dx, the end-point itself will increase by $a'(x)dx$.

Let us therefore return to Tobin. The objective is to choose L so as to maximize expected profits, with a FOC, derived from using Leibniz' formula on the last term, when we also use that $dF(x) = f(x)dx$ or $F'(x) = f(x)$:

$\frac{\partial \Pi}{\partial L} = P'(L) - r(1-k)D_0\frac{\partial y}{\partial L} - b(1-k)D_0\left[F(y)\frac{\partial y}{\partial L} + yf(y)\frac{\partial y}{\partial L}\right] + b(1-k)D_0 yf(y)\frac{\partial y}{\partial L} = 0.$

Moreover, using $\frac{\partial y}{\partial L} = \frac{1}{(1-k)D_0}$ and collecting terms, the volume of loans that maximizes expected profits, $\tilde{L}$, is determined from the FOC:

$\frac{\partial \Pi}{\partial L} = P'(\tilde{L}) - r - bF(y) = 0$

or $P'(\tilde{L}) = r + bF(y)$.

The condition amounts to state that the expected marginal revenue from making loans should be equal to marginal expected opportunity cost of lending (remember $F(y) = Pr(X < y)$, is the probability that the opportunity cost is $r + b$). Because F is increasing, this additional term is higher, the higher is L, as y itself is increasing in L. The marginal opportunity cost is therefore increasing in L. This means that if f goes to zero as x goes to infinity, this expected marginal cost of lending is illustrated in Figure 0.3.

Here the dashed curve is $r + bF(y)$; the curve starts at r and approaches $r + b$ as L gets larger.

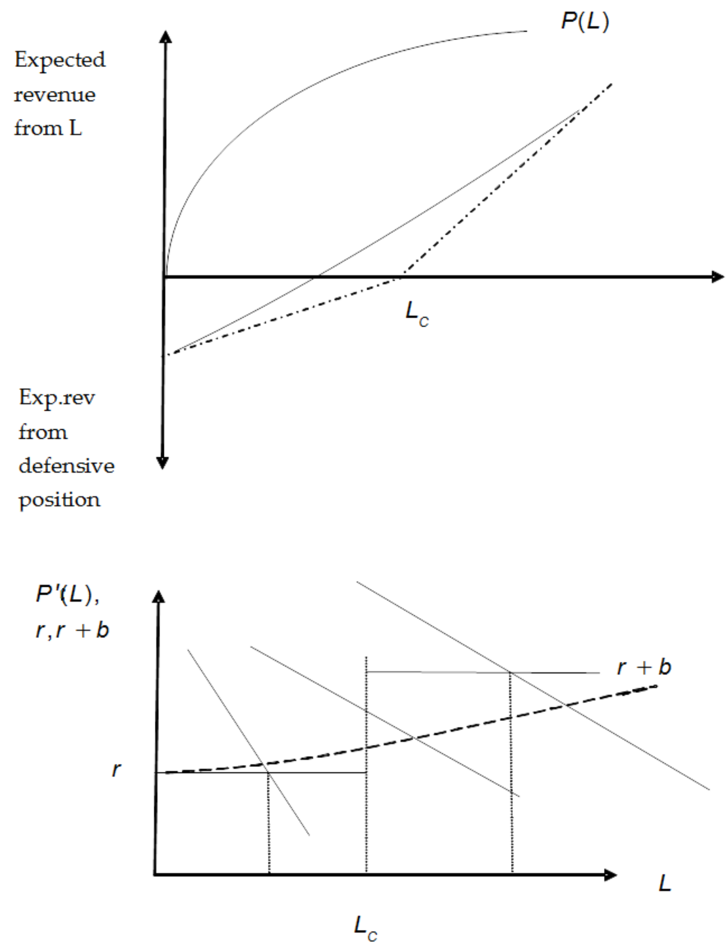


FIGURE 0.3. Revenue and cost of loans for random deposits (above). Marginal revenue and marginal cost of loans for random deposits (below).

Comparative Statics

1. Exogenous Change in Expected Deposits

Let's consider the bank response to an exogenous decrease in expected deposits D_0 say by 10%. As a first approximation, let's assume that $P'(L)$ is a constant: the marginal revenue from loans does not depend on the size of the loans. In this case, the bank will change L so that y is ultimately left unchanged. As $y = \frac{L-E-(1-k)D_0}{(1-k)D_0}$, if D_0 decreases by 10%, then also $L - E$ will decrease by 10%. If instead $P'(L)$ is not constant, and more specifically, it rises as L is reduced ($P''(L) < 0$), then the bank would reduce $L - E$ by LESS than 10%.

2. Change in the Yield of Defensive Assets

If r increases, the marginal opportunity cost of loans will increase: the bank will substitute defensive assets for loans and investments.

3. Change in the Required Reserve Ratio

If $P'(L)$ is a constant, the bank will adjust its loans as a result of a change in the required reserve ratio in a way that leaves the probability of a negative defensive position unchanged. If instead $P'(L)$ is not constant, and more specifically, it rises as L is reduced ($P''(L) < 0$), then the bank would reduce L by an amount that results in a threshold y larger than the threshold before the change in the required reserves ratio.