# Problem Set 1

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1. Population $= 1$

Friction: liquidity preference, private

$$u(c) = \frac{1}{1-\gamma} c^{1-\gamma}, \quad \gamma > 1$$

CRRA, Arrow-Pratt RRA

$$-\frac{c u''}{u'} = \gamma$$

$0 < \delta < 1$: "inalienated human capital"
"asymmetric information"

A) "social planner's problem": benchmark
"Pareto optimal"

$$\begin{array}{lcc} & \pi & 1-\pi \\ \text{population:} & \text{imp.} & \text{pat.} \end{array}$$

$$\max \quad \pi u(c_1) + (1-\pi) u(c_2)$$

s.t.

$$\pi c_1 = s \longrightarrow \text{storage}$$

$$(1-\pi) c_2 = (1-s) R$$

F.o.C.

$$\mathcal{L} = \pi u(c_1) + (1-\pi) u(c_2) - \lambda \left[ (1-\pi) c_2 - (1-\pi c_1) R \right]$$

$$\left.\begin{aligned}
\frac{\partial L}{\partial c_1} &= \pi u'(c_1) - \lambda \pi R = 0 \\
\frac{\partial L}{\partial c_2} &= (1-\pi) u'(c_2) - \lambda (1-\pi) = 0 \\
\frac{\partial L}{\partial \lambda} &= (1-\pi) c_2 - (1-\pi c_1) R = 0
\end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned}
u'(c_1) &= R u'(c_2) \\
u(c) &= \frac{c^{1-\gamma}}{1-\gamma}
\end{aligned}\right\} \Rightarrow$$

$$c_1^{-\gamma} = R c_2^{-\gamma}$$

$$C_1 = R C_2$$

$$\left.\begin{array}{l} \dfrac{C_2}{C_1} = R^{\frac{1}{\gamma}} \\ \text{constraints} \quad (1-\pi) C_2 = (1 - \pi C_1) R \end{array}\right\} \Rightarrow$$

$$C_1^* = \frac{R}{(1-\pi) R^{\frac{1}{\gamma}} + \pi R}$$

$$C_2^* = \frac{R^{1+\frac{1}{\gamma}}}{(1-\pi) R^{\frac{1}{\gamma}} + \pi R}$$

2. Properties of $(C_1^*, C_2^*)$

Claim: $1 < C_1^* < C_2^* < R$.

Proof: $\left.\begin{array}{l} u'(C_1^*) = R u'(C_2^*) \\ R > 1 \end{array}\right\}$

$$\left.\begin{array}{l} \Rightarrow \quad u'(C_1^*) > u'(C_2^*) \\ u(c) \Rightarrow u''(c) < 0 \end{array}\right\} \Rightarrow$$

concave

$$C_1^* < C_2^*.$$

Resource constraints: $(1-\pi) C_2^* = (1 - \pi C_1^*) R$

$$C_2^* = \frac{1 - \pi C_1^*}{1 - \pi} R$$

Possibilities:

a) $\begin{cases} C_1^* > 1 \Rightarrow \dfrac{1 - \pi C_1^*}{1-\pi} < 1 \\ C_2^* < R \end{cases}$

b) $\begin{cases} C_1^* \le 1 \Rightarrow \dfrac{1 - \pi C_1^*}{1-\pi} \ge 1 \\ C_2^* \ge R \end{cases}$

If a) is correct:

$$\gamma > 1 \Leftarrow \gamma = - \frac{c u''(c)}{u'(c)} > 1$$

$$\gamma > 1 \Leftarrow \gamma = \left( - \frac{c u''(c)}{u'(c)} > 1 \right)$$

$$c u''(c) + u'(c) < 0$$

$$\Rightarrow \frac{\partial [c u'(c)]}{\partial c} < 0 \quad (1)$$

a) $\begin{cases} c_1^* > 1 \\ c_2^* < R \end{cases}$ $\overset{(1)}{\Rightarrow} 1 \cdot u'(1) > c_1^* u'(c_1^*)$

$c_1^* < c_2^* \overset{(1)}{\Longrightarrow} c_1^* u'(c_1^*) > c_2^* u'(c_2^*)$

$$\Rightarrow 1 \cdot u'(1) > c_1^* u'(c_1^*) > c_2^* u'(c_2^*)$$

$$c_2^* < R \overset{(1)}{\Longrightarrow} c_2^* u'(c_2^*) > R u'(R)$$

$$\Rightarrow 1 \cdot u'(1) > c_1^* u'(c_1^*) > c_2^* u'(c_2^*) > R u'(R)$$

$$\overset{(1)}{\Rightarrow} 1 < R$$ ✓ fits the assumption

consistent $\Rightarrow$ a) ✓

b) $\begin{cases} c_1^* \leq 1 \\ c_2^* \geq R \end{cases}$

$R > 1$

$$u'(c_1^*) = R u'(c_2^*) \leq R u'(R)$$

$$c_2^* \geq R$$

$$u'(c_2^*) \leq u'(R)$$

$$R u'(R) < 1 \cdot u'(1)$$

$\overset{(1)}{\Rightarrow}$

$$\Rightarrow u'(c_1^*) < 1 \cdot u'(1)$$

concave

$$\Rightarrow c_1^* > 1 \Longleftrightarrow c_1^* \leq 1$$

concave

$\Rightarrow \quad \underline{c_1^* > 1} \quad \Longleftrightarrow \quad c_1^* \leq 1$

contradiction!

(b) ✗ q.e.d.

$\Rightarrow \quad 1 < \underline{c_1^*} < \underline{c_2^*} < R$

social planner: friction

① maximize social return

→ $s \uparrow$ with $R$ → $c \uparrow$

(without friction: $S = 1$ for $R$)

② friction: $s \downarrow$ → $\underline{\underline{c_1^*}}$ (liquidity insurance)

$c_1^* < c_2^*$ : "liquidity premium"

Social planner:

$t = 0$

| A | L |
|---|---|
| $1 - S$ | "deposit" $1$ |
| $S$ | |

"short assets" ($1 - S$)

"long assets" ($S$)

$S$ → "saving"

$t = 1$

| A | L |
|---|---|
| $1 - S$ → | $\pi c_1^*$ |
| $S R$ → | $(1 - \pi) c_2^*$ |

$t = 2$

| A | L |
|---|---|
| $S R$ | $(1 - \pi) c_2^*$ |

— no liquidation

Proof: liquidation is not optimal.

Suppose there's an optimal solution $(\tilde{c}_1, \tilde{c}_2)$

with liquidation.

$$\pi \tilde{c}_1 = \overset{\text{storage}}{S} + \overset{\text{from long assets}}{l\delta}$$

$$(1-\pi)\tilde{c}_2 = (1-S-l)R$$

$$\Rightarrow \quad \tilde{c}_1 = \frac{S + l\delta}{\pi}$$

However: if you invest $l$ in short assets

at $t=0$

$$c_1 = \frac{S + l}{\pi} \underset{\delta < 1}{>} \tilde{c}_1$$

$\Rightarrow$ contradiction that $(\tilde{c}_1, \tilde{c}_2)$ is optimal!

$\Rightarrow$ optimal solution doesn't include liquidation.

q.e.d.

3. $\rightarrow \gamma \rightarrow +\infty$: extreme risk averse

$$\lim_{\gamma \to +\infty} \frac{c_2^*}{c_1^*} = R^{\frac{1}{\gamma}} = R^0 = 1$$

$\Rightarrow c_2^* = c_1^*$ ← full insurance

(B) Autarky

$t = 0$ everyone has 1

→ invest $0 \leq \alpha \leq 1$ on short

$0 \leq 1-\alpha \leq 1$ on long.

$t = 1$ if impatient $\delta < 1$

$$C_1^a = \alpha + \underbrace{(1-\alpha)\delta}_{\text{liquidate}} \leq 1$$

$t = 2$ if patient $0 \leq \alpha \leq 1$

$$C_2^a = \underline{\alpha + (1-\alpha)R} \leq R$$

comp: $1 < C_1^* < C_2^* < R$

$$(C_1^a, C_2^a) < (C_1^*, C_2^*)$$

(c) Bond market : trade

$t = 0$ $0 \leq \alpha \leq 1$ short

$0 \leq 1-\alpha \leq 1$ long

$t = 1$ impatient : sell. at price

$$\underline{C_1^b} = \alpha + \underbrace{(1-\alpha)R}_{\text{future}}\,\underline{b}$$

$t = 2$ patient

$$\underline{C_2^b} = \frac{\alpha}{b} + (1-\alpha)R$$

proceeds from $t=1$ bought from impatient

if $\underline{b > \frac{1}{R}}$ ✗

$\Rightarrow R \cdot b > 1 \Rightarrow \underline{\alpha = 0}$, no storage

no market !

if $b < \frac{1}{R}$ ✗

$\Rightarrow R \cdot b < 1 \Rightarrow \alpha = 1$, no long assets !

no market !

$\Rightarrow b = \frac{1}{R}$, with $R \cdot b = 1$

$\Rightarrow \quad 0 < \alpha < 1$

$$C_1^b = \alpha + (1-\alpha)\frac{R \cdot b}{1} = 1$$

$$C_2^b = \underbrace{\frac{\alpha}{b}}_{=\frac{1}{R}} + (1-\alpha) R = R$$

$$(C_1^b, C_2^b) = (1, R) \neq (C_1^*, C_2^*)$$

(0) Banks $t=0$ $d_0 = (C_1^*, C_2^*)$

a) do is optimal ✓

b) do is feasible. $\Leftarrow$ resource constraints

c) ~~participation constraints~~ incentive compatible $\Leftarrow$ type is private information

{ willing to join do
{ mimic the others: no

P.C. $\Pi^{Bank} \geq 0$

consumers

I.C. imp.

par. $1 < C_1^* < C_2^* < R$

2, socially optimal $\begin{cases} C_1^a \leq 1 \\ C_2^a \leq R \end{cases}$

2. Claim: Bank run solution is another N.E.

Proof: N.E. Best response $\rightarrow$ no profitable deviation

In a bank run:

demand: $d = c_1^* \times 1$

supply: $\underline{S}^{max} =$ short assets + liquidate long

$= \pi c_1^* + (1 - \pi c_1^*) \delta$ $\frac{\delta}{c_1}$

$< c_1^*$

$d > \max S$

$\rightarrow \quad c_1^r = \dfrac{\max S}{1} = \pi c_1^* + (1 - \pi c_1^*)\delta < c_1^*$

$c_2^r = 0$

<u>deviation?</u> wait till $t = 2$ $\quad c_2^r = 0 < \underline{c_1^r}$

$\rightarrow$ N.E. $\qquad$ q.e.d.

What if we drop $\underline{\delta < 1}$ ?

if run is not N.E. $\rightarrow$ show deviation is profitable,

wait till $t = 2$ and $c_2 > \underline{c_1^r}$

$\rightarrow \quad S^{max} > d$

$\rightarrow \quad \pi c_1^* + (1 - \pi c_1^*)\delta > c_1^*$

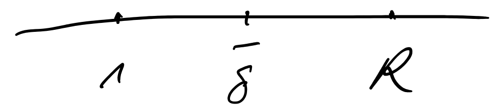
$\rightarrow \quad \delta > \dfrac{(1-\pi)c_1^*}{1 - \pi c_1^*} = \bar{\delta}$

$> \dfrac{(1-\pi)c_1^*}{c_1^* - \pi c_1^*} = 1$

$$c_1^* - \pi c_1$$

$$\underline{c_1^*} > 1$$

$$\Rightarrow \quad \delta > \bar{\delta} > 1$$




B)

| Mechanism | Reality |
| --- | --- |
| 1. <u>deposit insurance</u> | credit risk → <u>moral hazard</u><br>⇒ shift risk<br>→ no full ins. |
| 2. bank holiday<br>$S_1 = \pi C_1^*$ | $\underline{\pi}$ is not observable |
| 3. <u>interbank</u> lending | <u>"credit risk"</u><br>→ "<u>hair cut</u>" |
| 4. C.B. LoLR / ELA | <u>c.r.</u> ← big bank<br><u>"too-big-to-fail"</u> |

c) asset purchase → $\delta \uparrow > \bar{\delta}$

c) asset purchase $\rightarrow$ $\underline{\delta}$ | $> \delta$