ECON 4335 Economics of Banking, Fall 2017
Problem Set 6: Solution

November 2, 2017

Overborrowing, externalities, and systemic risk

Consider a small open economy with a tradable goods sector and a nontradable goods sector. Only tradable goods can be traded internationally; nontradable goods have to be consumed domestically. The economy is populated by a continuum of identical households of measure one, living for 2 periods $t = 0, 1$, *with preferences given by* $u(c) = \ln c_0^T + \ln c_0^N + \ln c_1^T$, *in which* c_0^T, c_0^N, c_1^T *are consumption of tradable goods at* $t = 0$, *consumption of nontradable goods at* $t = 0$, *consumption of tradable goods at* $t = 1$, *respectively.*

A representative household starts with initial asset b_0 *at* $t = 0$, *and ends after* $t = 1$ *with zero asset, i.e.,* $b_2 = 0$. *Note that* b_0 *can be positive or negative: when* $b_0 < 0$, *the household starts with initial debt. The timeline of events is as follows:*

- *At* $t = 0$, *the representative household receives both an endowment of tradable goods* y_0^T *and an endowment of nontradable goods* y_0^N *for consumption.* y_0^T *is a random variable that is drawn from a distribution with cummulative distribution function* $F(y)$, *while* y_0^N *is constant. After* $\left(y_0^T, y_0^N\right)$ *is revealed, the household can also borrow from abroad by purchasing a one-period, non-state contingent foreign bond denominated in units of tradables that demands a fixed interest rate* r *— normalized to be 0, determined exogenously in the world market. Normalize the price of tradables to 1 and denote the price of nontradable goods by* p_0^N. *In addition, the household's debt is securitized such that its total debt cannot exceed a fraction* $0 < \kappa < 1$ *of its total income from tradables and nontradables;*
- *At* $t = 1$, *starting with total asset* b_1 *the representative household only receives an endowment of tradable goods* y_1^T *for consumption.* y_1^T *is constant.*

1. *Specify the representative household's budget constraints, borrowing constraint, and lifetime optimization problem.*

The household's optimization problem is

$$\max_{\{c_0^T, c_0^N, b_1, c_1^T\}} \quad u = \ln c_0^T + \ln c_0^N + \ln c_1^T,$$

$$\begin{aligned} s.t. \quad & c_0^T + p_0^N c_0^N + b_1 = b_0 + y_0^T + p_0^N y_0^N, \\ & c_1^T = b_1 + y_1^T, \\ & b_1 \geq -\kappa \left(y_0^T + p_0^N y_0^N \right). \end{aligned}$$

2. *Compute the first order conditions for the household's optimization problem:*

(a) *Derive the first order conditions with respect to c_0^T and c_0^N, then determine p_0^N;*
Set up the Lagrangian for the optimization problem as

$$\begin{aligned} \mathcal{L} \;=\; & \ln c_0^T + \ln c_0^N + \ln c_1^T \\ & + \lambda_0 \left(b_0 + y_0^T + p_0^N y_0^N - c_0^T - p_0^N c_0^N - b_1 \right) \\ & + \lambda_1 \left(b_1 + y_1^T - c_1^T \right) \\ & + \nu \left[b_1 + \kappa \left(y_0^T + p_0^N y_0^N \right) \right], \end{aligned}$$

and the first order conditions with respect to c_0^T and c_0^N are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_0^T} &= \frac{1}{c_0^T} - \lambda_0 = 0; \\ \frac{\partial \mathcal{L}}{\partial c_0^N} &= \frac{1}{c_0^N} - \lambda_0 p_0^N = 0. \end{aligned}$$

Combine to get

$$\frac{c_0^T}{c_0^N} = p_0^N.$$

As non-tradables are not used for bond transactions, they have to be consumed, i.e., $c_0^N = y_0^N$, and $\frac{c_0^T}{y_0^N} = p_0^N$.

(b) *Derive the first order conditions with respect to c_1^T and b_1, then determine the Euler equation. Why is the borrowing constraint* occasionally *binding?*
The first order conditions with respect to c_1^T and b_1 are

$$\frac{\partial \mathcal{L}}{\partial c_1^T} = \frac{1}{c_1^T} - \lambda_1 = 0;$$

$$\frac{\partial \mathcal{L}}{\partial b_1} = -\lambda_0 + \lambda_1 + \nu = 0.$$

Combining the results from before to get $\frac{1}{c_0^T} = \frac{1}{c_1^T} + \nu$. The value of ν is the shadow price of bonds, by Kuhn-Tucker condition,

$$\nu \begin{cases} = 0 & \text{if } b_1 > -\kappa\left(y_0^T + p_0^N y_0^N\right) \\ \geq & \text{if } b_1 = -\kappa\left(y_0^T + p_0^N y_0^N\right). \end{cases}$$

Therefore, if the borrowing constraint is not binding, $\nu = 0$ and $\frac{1}{c_0^T} = \frac{1}{c_1^T}$ or $c_0^T = c_1^T$; if the borrowing constraint is binding, $\nu \geq 0$ and $\frac{1}{c_0^T} \geq \frac{1}{c_1^T}$ or $c_0^T \leq c_1^T$.

3. *Determine c_0^T:*

   (a) *Under what condition(s) is the borrowing constraint not binding? In this case, use the results from exercise 2(b) to determine c_0^T;*
   If the borrowing constraint is not binding, $c_0^T = c_1^T$. Combining budget constraints $c_0^T + p_0^N c_0^N + b_1 = b_0 + y_0^T + p_0^N y_0^N$ and $c_1^T = b_1 + y_1^T$ and using the fact that $c_0^N = y_0^N$, one can get $c_0^T = \frac{b_0 + y_0^T + y_1^T}{2}$. That the borrowing constraint is not binding means $b_1 > -\kappa\left(y_0^T + p_0^N y_0^N\right)$, combing the budget constraints and the value of c_0^T

   $$b_1 = b_0 + y_0^T - c_0^T > -\kappa\left(y_0^T + p_0^N y_0^N\right) = -\kappa\left(y_0^T + c_0^T\right),$$

   rearrange to get

   $$b_0 > \frac{-(3\kappa + 1)y_0^T + (1 - \kappa)y_1^T}{1 + \kappa}.$$

   That is, the borrowing constraint is not binding if the household does not start with too much debt.

   (b) *When the borrowing constraint is binding, compute c_0^T.*
   When the borrowing constraint is binding, combine the binding borrowing constraint with the budget constraints and get $c_0^T = \frac{b_0 + (1+\kappa)y_0^T}{1-\kappa}$. In this case, $b_1 = b_0 + y_0^T - c_0^T = -\frac{\kappa b_0 + 2\kappa y_0^T}{1-\kappa}$.

4. *Consider two situations at $t = 0$: the economy can be either in normal state $y_0^T = \bar{y}$, i.e., the household receives a mean value $\bar{y}$, or crisis state $y_0^T = \bar{y} - 1$, i.e., y_0^T is below the mean. Suppose the household knows the true state before it borrows.*

   (a) *If borrowing constraint is not binding in both states, how does c_0^T react to the crisis,*

compared with c_0^T in the normal state?
Use the result from exercise 3(a), $c_0^T(\overline{y}) - c_0^T(\overline{y}-1) = \frac{1}{2}$.

(b) *If borrowing constraint is binding in both states, how does c_0^T react to the crisis, compared with c_0^T in the normal state?*
Use the result from exercise 3(b), $c_0^T(\overline{y}) - c_0^T(\overline{y}-1) = \frac{1+\kappa}{1-\kappa} > 1 > \frac{1}{2}$.

☞Bianchi, J. (2011), Overborrowing and systemic externalities in the business cycle, *American Economic Review* 101, 3400-3426.