Econ 4335 Exam Solution: Macro

Problem 1. Overborrowing

Consider a small open economy with a tradable goods sector and a nontradable goods sector. Only tradable goods can be traded internationally; nontradable goods have to be consumed domestically. The economy is populated by a continuum of identical households of measure one, living for 2 periods $t = 0, 1$, with preferences given by $u(c) = lnc_0^T + lnc_0^N + lnc_1^T$, in which c_0^T, c_0^N, c_1^T are consumption of tradable goods at $t = 0$, consumption of nontradable goods at $t = 0$, consumption of tradable goods at $t = 1$, respectively.

A representative household starts with initial asset b_0 at $t = 0$, and ends after $t = 1$ with zero asset, i.e., $b_2 = 0$. Note that b_0 can be positive or negative: when $b_0 < 0$, the household starts with initial debt. The timeline of events is as follows:

- At $t = 0$, the representative household receives both an endowment of tradable goods y_0^T and an endowment of nontradable goods y_0^N for consumption. Both y_0^T and y_0^N can be random variables drawn from certain distributions. After $\left(y_0^T, y_0^N\right)$ is revealed, the household can save ($b_1 > 0$) or borrow ($b_1 < 0$) from abroad by purchasing a one-period, non-state contingent foreign bond denominated in units of tradables. Interest rate r is normalized to be 0, determined exogenously in the world market. Normalize the price of tradables to 1 and denote the price of nontradable goods by p_0^N. In addition, the household's debt is securitized such that its total debt cannot exceed a fraction $0 < \kappa < 1$ of its total income from tradables and nontradables;

- At $t = 1$, starting with total asset b_1 the representative household only receives an endowment of tradable goods y_1^T for consumption. y_1^T is constant.

1. Specify the representative household's budget constraints, borrowing constraint, and lifetime optimization problem.

$$\max_{c_0^T, c_0^N, b_1, c_1^T} \log\left(c_0^T c_0^N\right) + \log c_1^T$$

$$\text{s.t. } c_0^T + p_0^N c_0^N + b_1 = b_0 + y_0^T + p_0^N y_0^N, \tag{1}$$

$$c_1^T = b_1 + y_1^T, \tag{2}$$

$$b_1 \geq -\kappa\left(y_0^T + p_0^N y_0^N\right). \tag{3}$$

2. Compute the first order conditions for the household's optimization problem:

   (a) Derive the first order conditions with respect to c_0^T and c_0^N, then determine p_0^N;

   Form the Lagrangian:

$$\begin{aligned} L = & \log c_0^T + \log c_0^N + \log c_1^T \\ & + \lambda_0\left(b_0 + y_0^T + y_0^N - c_0^T - p_0^N c_0^N - b_1\right) \\ & + \lambda_1\left(b_1 + y_1^T - c_1^T\right) \\ & + \nu\left(b_1 + \kappa\left(y_0^T + p_0^N y_0^N\right)\right). \end{aligned}$$

   Derive the FOCs:

$$\frac{\partial L}{\partial c_0^T} : \frac{1}{c_0^T} - \lambda_0 = 0, \tag{4}$$

$$\frac{\partial L}{\partial c_0^N} : \frac{1}{c_0^N} - \lambda_0 p_0^N = 0.$$

Combine these two equations:

$$\begin{aligned}\frac{1}{c_0^N} &= \lambda_0 p_0^N = p_0^N \frac{1}{c_0^T} \Rightarrow \\ \frac{c_0^T}{c_0^N} &= p_0^N.\end{aligned}$$

Substitute the market clearing condition for non-tradable goods, i.e., $c_0^N = y_0^N$:

$$p_0^N = \frac{c_0^T}{y_0^N}. \tag{5}$$

(b) Derive the first order conditions with respect to c_1^T and b_1, then determine the Euler equation. Why is the borrowing constraint occasionally binding?

FOCs:

$$\begin{aligned}\frac{\partial L}{\partial b_1} &: -\lambda_0 + \lambda_1 + \nu = 0, \\ \frac{\partial L}{\partial c_1^T} &: \frac{1}{c_1^T} - \lambda_1 = 0.\end{aligned}$$

Combine these two equations with equation (4):

$$\begin{aligned}-\frac{1}{c_0^T} + \frac{1}{c_1^T} + \nu = 0 \Rightarrow \\ \frac{1}{c_0^T} = \frac{1}{c_1^T} + \nu.\end{aligned} \tag{6}$$

ν is the Lagrangian multiplier for the borrowing constraint - equation (3), so whether it is positive or zero depends on whether the constraint is binding or not. The Kuhn-Tucker condition tells us the following. (1) If constraint (3) is not binding, then

$$\begin{aligned}& b_1 > -\kappa\left(y_0^T + p_0^N y_0^N\right), \\ & \nu = 0.\end{aligned}$$

Substitute ν into (6):

$$\frac{1}{c_0^T} = \frac{1}{c_1^T}. \tag{7}$$

(2) If constraint (3) is binding, then

$$b_1 = -\kappa \left(y_0^T + p_0^N y_0^N \right), \tag{8}$$

$$\nu \geq 0 \Rightarrow$$

$$\frac{1}{c_0^T} \geq \frac{1}{c_1^T}.$$

3. Determine c_0^T:

   (a) Under what condition(s) is the borrowing constraint not binding? In this case, use the results from exercise 2(b) to determine c_0^T;

   Substitute unconstrained Euler equation (7) into budget constraints, i.e., equation (1) and (2), and express c_1^T and b_1 using c_0^T we get:

$$c_0^T + p_0^N c_0^N + c_0^T - y_1^T = b_0 + y_0^T + p_0^N y_0^N \Rightarrow$$

$$c_0^T = \frac{b_0 + y_0^T + y_1^T}{2}. \tag{9}$$

   Notice that $p_0^N c_0^N$ always cancels $p_0^N y_0^N$.

   Furthermore, given c_0^T, we can compute b_1 and can check under which condition the borrowing constraint is not binding.

$$
\begin{aligned}
b_1 = b_0 + y_0^T - c_0^T > -\kappa \left(y_0^T + p_0^N y_0^N\right) &= -\kappa \left(y_0^T + c_0^T\right) \Rightarrow \\
b_0 + (1+\kappa)\, y_0^T - (1-\kappa)\, c_0^T &> 0 \Rightarrow \\
b_0 + (1+\kappa)\, y_0^T - (1-\kappa) \frac{b_0 + y_0^T + y_1^T}{2} &> 0 \Rightarrow \\
2b_0 + 2(1+\kappa)\, y_0^T - (1-\kappa)\left(b_0 + y_0^T + y_1^T\right) &> 0 \Rightarrow \\
(1+\kappa)\, b_0 + (1+3\kappa)\, y_0^T &> (1-\kappa)\, y_1^T.
\end{aligned}
$$

We can see that if b_0 and y_0^T are sufficiently high, the borrowing constraint is not binding. In other words, given y_0^T, if b_0 is high enough, i.e.,

$$
b_0 > \frac{-(1+3\kappa)\, y_0^T + (1-\kappa)\, y_1^T}{1+\kappa},
$$

then the borrowing constraint is not binding.

(b) When the borrowing constraint is binding, compute c_0^T.

Substitute equation (8) into equation (1):

$$
\begin{aligned}
c_0^T - \kappa \left(y_0^T + p_0^N y_0^N\right) &= b_0 + y_0^T \Rightarrow \\
c_0^T - \kappa \left(y_0^T + c_0^T\right) &= b_0 + y_0^T \Rightarrow \\
c_0^T &= \frac{b_0 + (1+\kappa)\, y_0^T}{1-\kappa}.
\end{aligned} \tag{10}
$$

We can also compute b_1:

$$
\begin{aligned}
b_1 &= b_0 + y_0^T - c_0^T \\
&= -\frac{\kappa b_0 + 2\kappa y_0^T}{1-\kappa}.
\end{aligned}
$$

4. Consider the case that the shocks only affect tradable goods: y_0^N is constant while the

realization of y_0^T can be high or low. The economy can be either in normal state $y_0^T = \bar{y}$, or crisis state with a lower tradable goods production: $y_0^T = \bar{y} - 1$. Remember that the household knows the true state before it borrows.

(a) If borrowing constraint is not binding in both states, how does c_0^T react to the crisis, compared with c_0^T in the normal state?

From equation 9, we get:

$$c_0^T(\bar{y}) - c_0^T(\bar{y} - 1) = \frac{1}{2}.$$

(b) If borrowing constraint is binding in both states, how does c_0^T react to the crisis, compared with c_0^T in the normal state?

From equation 10, we get:

$$c_0^T(\bar{y}) - c_0^T(\bar{y} - 1) = \frac{1+\kappa}{1-\kappa} > \frac{1}{2}.$$

In fact, it is even greater than 1.

5. Consider the case that the shocks only affect nontradable goods: y_0^T is constant while y_0^N can be high or low. In the normal state, $\left(y_0^T, y_0^N\right) = \left(y^T, y^N\right)$ and in the crisis state, $\left(y_0^T, y_0^N\right) = \left(y^T, y^N - 1\right)$. Does the shock to non-tradable goods affect the borrowing constraint and the consumption of tradable goods c_0^T? Why? (You may answer this question by solving the problem analytically to or by simply arguing with logic. In the latter case, the equations for borrowing constraint and non-tradable goods price p_0^N solved above are useful.)

One can answer the question by solving for the equilibrium. In the first case that the constraint is not binding (b_0 large enough), the equilibrium is the following. The tradable good consumption does not depend on y_0^N:

$$c_0^T = \frac{b_0 + y_0^T + y_1^T}{2}$$
$$= \frac{b_0 + y^T + y_1^T}{2}.$$

The non-tradable goods price depends on the realization of y_0^N but its value $p_0^N y_0^N$ does not:

$$p_0^N = \frac{c_0^T}{y_0^N} \Rightarrow$$
$$p_0^N y_0^N = c_0^T.$$

So the borrowing constraint does not depend on y_0^N:

$$b_1 \geq -\kappa \left(y_0^T + p_0^N y_0^N \right)$$
$$= -\kappa \frac{b_0 + 3y^T + y_1^T}{2}.$$

Similar for the case that the constraint is binding (b_0 small enough). As computed in Equation 10, c_0^T depends on b_0 and y_0^T:

$$c_0^T = \frac{b_0 + (1+\kappa)\, y^T}{1-\kappa}.$$

Then $p_0^N y_0^N$ also depends only on c_0^T but not y_0^N. This implies that b_1 does not depend on y_0^N.

$$p_0^N y_0^N = c_0^T,$$

$$
\begin{aligned}
b_1 &= -\kappa \left(y_0^T + p_0^N y_0^N \right) \\
&= -\kappa \left(y^T + c_0^T \right) .
\end{aligned}
$$

One can also answer this question by just stating the following logic. From the equation for the non-tradable goods price, we know that the total value of non-tradable goods does not depend on the realization of non-tradable goods. When the realization is low - y_0^N small - the prices goes up, so the total value does not change. The total value of the non-tradable goods is what matters for the borrowing constraint, so the borrowing constraint does not change, and the household consumption of tradable goods also does not change.