Problem 1

Consider a risk-averse person who maximizes expected utility. Show that in a simple portfolio problem with one risky and one risk free asset, the person will invest a strictly positive amount in the risky asset if and only if its expected rate of return, $E(\tilde{r})$, exceeds the rate of return on the risk free asset, r_f. Give a verbal interpretation of the result.

Answer to problem 1

This follows lecture notes from 25 August 2009, pp. 4–6, see separate pages which follow below. The bottom half of p. 4 is not relevant. The formulation with discrete probability distribution for $\tilde{r}$ is sufficient to answer the problem, so Leibniz' formula is not necessary. The second-order condition (top of p. 6) is not very important here, since the argument in the rest of p. 6 relies on a local property of $W(a)$ around $a = 0$ (although a global maximum might, of course, appear in a weird place if the function had not been everywhere concave). The verbal interpretation is that risk averse individuals need to be paid (in the sense of higher expected values) in order to take on risk, but that a slight payment is sufficient to induce some (slight) risk taking. This is known as local risk neutrality, since the risk plays no role in this marginal decision.

Problem 2

Consider an economy where the standard Capital Asset Pricing Model holds, with a large number of different shares traded in the stock market. You are not asked to derive the model or even state all assumptions behind it. Consider two of the many firms in this economy, and assume that they do not pay any dividends in the period we are concerned with. For simplicity we assume they are financed by equity only, no debt. The total value today of all shares in the two firms are X_0 and Y_0, respectively, and their respective total share values one period into the future are $\tilde{X}_1$ and $\tilde{Y}_1$, both stochastic. The ratio of the total values today is $X_0/Y_0 = 1/5$.

(a)

A merger between the firms is considered. Assume that the merged firm will have total value next period equal to $\tilde{Z}_1 = \tilde{X}_1 + \tilde{Y}_1$. Show how today's value of the merged firm, Z_0, relates to X_0 and Y_0. Show how the beta of the shares in the merged firm, β_Z, relates to the betas of the two existing firms, β_X and β_Y.

Answer to problem 2(a)

The value of the merged firm is

$$Z_0 = \frac{1}{1+r_f}\left[E(\tilde{Z}_1) - \lambda \operatorname{cov}(\tilde{Z}_1, \tilde{r}_m)\right],$$

where $\lambda \equiv [E(\tilde{r}_m) - r_f]/\operatorname{var}(\tilde{r}_m)$, $\tilde{r}_m$ is the rate of return on the market portfolio, and r_f is the risk free interest rate. This can be rewritten as

$$Z_0 = \frac{1}{1+r_f}\left[E(\tilde{X}_1 + \tilde{Y}_1) - \lambda \operatorname{cov}(\tilde{X}_1 + \tilde{Y}_1, \tilde{r}_m)\right].$$

Using the additivity of expectation and covariance, we find that $Z_0 = X_0 + Y_0$.

The beta of the merged firm is $\operatorname{cov}((\tilde{Z}_1/Z_0)-1, \tilde{r}_m)/\operatorname{var}(\tilde{r}_m) = \operatorname{cov}(\tilde{Z}_1/Z_0, \tilde{r}_m)/\operatorname{var}(\tilde{r}_m)$. This turns out to be a value-weighted average of the betas of the two previous firms. The value weights are

$$\frac{X_0}{Z_0} = \frac{X_0}{X_0 + 5X_0} = \frac{1}{6} \quad \text{and} \quad \frac{Y_0}{Z_0} = \frac{5X_0}{X_0 + 5X_0} = \frac{5}{6}.$$

The value-weighted average holds for the covariances, as well:

$$\operatorname{cov}\left(\frac{\tilde{Z}_1}{Z_0}, \tilde{r}_m\right) = \operatorname{cov}\left(\frac{\tilde{X}_1 + \tilde{Y}_1}{Z_0}, \tilde{r}_m\right) = \operatorname{cov}\left(\frac{\tilde{X}_1}{Z_0}, \tilde{r}_m\right) + \operatorname{cov}\left(\frac{\tilde{Y}_1}{Z_0}, \tilde{r}_m\right) =$$

$$= \frac{X_0}{Z_0}\operatorname{cov}\left(\frac{\tilde{X}_1}{X_0}, \tilde{r}_m\right) + \frac{Y_0}{Z_0}\operatorname{cov}\left(\frac{\tilde{Y}_1}{Y_0}, \tilde{r}_m\right).$$

Divide both sides of the equation with $\operatorname{var}(\tilde{r}_m)$, plug in the weights, and find

$$\beta_Z = \frac{1}{6}\beta_X + \frac{5}{6}\beta_Y.$$

(b)

The rates of return of the shares of the firms are $\tilde{r}_X$ and $\tilde{r}_Y$, respectively, with the properties that $E(\tilde{r}_X) = 0.04$, $E(\tilde{r}_Y) = 0.16$, $\text{var}(\tilde{r}_X) = 0.09 = 0.3^2$, $\text{var}(\tilde{r}_Y) = 0.16 = 0.4^2$, and $\text{cov}(\tilde{r}_X, \tilde{r}_Y) = 0.02$. Assuming that the merger does not happen: Show that the minimal variance of the rate of return of any possible portfolio of these two shares is $0.6/9 \approx 0.06667 \approx 0.2582^2$. Illustrate with a suitable diagram which portfolios can be created from the two shares. Assuming instead that the merger happens, show the location of the merged firm's shares in the diagram.

Answer to problem 2(b)

The minimization of the variance of a portfolio of two risky assets is found in lecture notes of 1 September 2009, p. 6. With the notation there (1 there is X here, 2 there is Y here, a is portfolio weight for X):

$$\sigma_p^2 = a^2\sigma_1^2 + (1-a)^2\sigma_2^2 + 2a(1-a)\sigma_{12}.$$

First-order conditions for value of a at the minimum-variance point:

$$0 = \frac{d\sigma^2}{da} = 2a\sigma_1^2 - 2(1-a)\sigma_2^2 + (2-4a)\sigma_{12}$$

gives

$$a = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} \equiv a_{\min}.$$

For the given numbers we find

$$a_{\min} = \frac{0.4^2 - 0.02}{0.3^2 + 0.4^2 - 2 \cdot 0.02} = \frac{2}{3}.$$

Plugging into the variance formula:

$$\sigma_p^2 = \frac{4}{9} \cdot 0.09 + \frac{1}{9} \cdot 0.16 + 2 \cdot \frac{2}{3}\frac{1}{3} \cdot 0.02 = \frac{0.6}{9}.$$

Observe that the expected rate of return at the minimum-variance point is $\frac{2}{3} \cdot 0.04 + \frac{1}{3} \cdot 0.16 = 0.08$.

The diagram (see separate page below) has standard deviation of the rate of return on the horizontal axis, expected rate of return on the vertical. It shows the hyperbola, which is the location of portfolios when $a \in [0, 1]$ (no short sales). Short selling would extend the hyperbola in both directions. The hyperbola is symmetrical around the horizontal line at 0.08. Thus the variance is the same for portfolio weight $a = 1$ (everything in X) and for portfolio weight $a = 1/3$ (which gives expected rate of return 0.12). This has consequences for the answer to (c) below.

The merged firm is a portfolio with weight $a = 1/6$. Thus its location is at the hyperbola for expected rate of return equal to 0.14.

(c)

Discussing the possible benefits and drawbacks of the merger, one person argues, "The return on the shares of the merged firm will have a lower variance than the shares of any of the two existing firms. This is a benefit for shareholders." Discuss both parts of this statement: What can you say about the first, factual claim? What can you say about the benefit for shareholders?

Answer to problem 2(c)

The first, factual claim is wrong. The rate of return $\tilde{r}_Z$ will have a higher variance than $\tilde{r}_X$, cf. the argument given in the answer to part (b) above. The second part about the benefit for shareholders is not true within the model, if "benefit" is interpreted as a situation they would strictly prefer. This is not even true for the shareholders of Y, who *will* experience a lower variance. The reason is that within the model, all shareholders hold diversified portfolios. The risky part of their portfolios is the same for all, namely the market portfolio, and this would not change as a result of the merger. In the model, this is true not only as an approximation, but exactly: The market portfolio has each share in proportion to its total market value. Thus the market portfolio already contains X and Y shares in the same proportion in which they will appear in the merged firm, and thus in the new market portfolio after the merger.

Problem 3

(a)

Show how to derive the value of a call option by an absence-of-arbitrage argument in a binomial model of share prices. Assume that the option is of European type with expiration one period into the future, and that the share does not pay dividends during that period. In this model the relative change in the share price between two periods is either u or d, while the riskless interest rate is r.

Answer to problem 3(a)

This is shown in lecture notes of 3 November 2009, pp. 4–6, see separate pages below. In Hull's book, which is on the reading list for the course, the derivation is instead based on buying Δ shares and selling (issuing, writing, shorting) one call option. It is shown how to choose Δ in such a way that this portfolio gives a non-stochastic outcome next period. Then it is argued that the rate of return on such a portfolio must be the risk free interest rate. The equation system to solve for Δ is the same as in the lecture notes. When f is the call option value, the cost of setting up the portfolio is $S_0\Delta - f$, and one can solve for $f = e^{-rT}[pf_u + (1-p)f_d]$. (There seems to be a slight formal difference between Hull's equation and those in the lecture notes, namely that Hull uses T in the discounting. However, his derivation is based on T being one period into the future.)

(b)

A variable defined as

$$p = \frac{e^r - d}{u - d}$$

has a particular meaning. Does this variable relate to the call option value, and if yes, how? Does this variable play a role in the absence-of-arbitrage argument, and if yes, how? Does this variable relate to the probability of some change in the share price, and if yes, how?

Answer to problem 3(b)

The variable p relates to the call option value in the sense that it simplifies the formula. After the formula has been derived, and p has been defined, one observes that

$$c = S\Delta + B = \frac{(c_u - c_d)e^r + uc_d - dc_u}{(u-d)e^r} = \frac{(e^r - d)c_u + (u - e^r)c_d}{(u-d)e^r}$$

can be rewritten as

$$c = \frac{pc_u + (1-p)c_d}{e^r}.$$

However, p played no role in the absence-of-arbitrage argument.

The probability $\Pr(S_1/S_0 = u)$ is p^* in this model. This is an exogenously given variable, which can take any value between 0 and 1, independently of p. However, since p is also between 0 and 1, it may be interpreted as a probability. In particular, we see that $E(c_1/c_0)$ would have been equal to e^r if the probability of u had been p, not p^*. In fact, the same is true for $E(S_1/S_0)$. Thus we observe that p is the value which p^* must have if the share price, the option price, and the value of a riskless bond all should have the same expected rate of return. This would have been true if all agents were risk neutral. This turns out to be another method for pricing options (at least of the European type, we have not discussed whether it works for the American type): Since the option value is independent of p^*, we might consider what it would be if $p^* = p$, but in that case we do not need the absence-of-arbitrage argument, we could just use the expectation given this p.

Risk aversion and simple portfolio problem

(Chapter 5 in Danthine and Donaldson.)

Simple portfolio problem, one risky, one risk free asset. Total investment is Y_0, a part of this, a, is invested in risky asset with rate of return $\tilde{r}$, while $Y_0 - a$ is invested at risk free rate r_f. Expected utility becomes a function of a, which the investor wants to maximize by choosing a:

$$W(a) \equiv E\{U[\tilde{Y}_1]\} \equiv E\{U[Y_0(1+r_f) + a(\tilde{r} - r_f)]\}, \tag{1}$$

based on $\tilde{Y}_1 = (Y_0 - a)(1 + r_f) + a(1 + \tilde{r})$.

Solution of course depends on investor's U function. Assuming $U'' < 0$ and interior solutions we can show:

- Optimal a strictly positive if and only if $E(\tilde{r}) > r_f$.
- When the optimal a is strictly positive:
  - Optimal a independent of Y_0 for CARA, increasing in Y_0 for DARA, decreasing in Y_0 for IARA.
  - (CARA means Constant absolute risk aversion, DARA means Decreasing ARA, IARA means Increasing ARA.)
  - Optimal a/Y_0 independent of Y_0 for CRRA, increasing in Y_0 for DRRA, decreasing in Y_0 for IRRA.
  - (CRRA, DRRA, IRRA refer to relative risk aversion instead of absolute.)

This gives a better understanding of what it means to have, e.g., decreasing absolute risk aversion.

First-order condition for simple portfolio problem

To find f.o.c. of maximization problem (1), need take partial derivative of expectation of something with respect to a deterministic variable. Straight forward when $\tilde{r}$ has discrete probability distribution, with π_θ the probability of outcome r_θ. Then $W(a) =$

$$E\{U[Y_0(1+r_f)+a(\tilde{r}-r_f)]\} = \sum_\theta \pi_\theta U[Y_0(1+r_f)+a(r_\theta-r_f)],$$

and the f.o.c. with respect to a is

$$\begin{aligned} W'(a) &= \sum_\theta \pi_\theta U'[Y_0(1+r_f)+a(r_\theta-r_f)](r_\theta-r_f) \\ &= E\{U'[Y_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)\} = 0. \qquad (2) \end{aligned}$$

The final equation above, (2), is also f.o.c. when distribution continuous, cf. Leibniz' formula (see Sydsæter et al): The derivative of a definite integral (with respect to some variable other than the integration variable) is equal to the definite integral of the derivative of the integrand.

Observe that in (2) there is the expectation of a product, and that the two factors $U'[Y_0(1+r_f)+a(\tilde{r}-r_f)]$ and $(\tilde{r}-r_f)$ are not stochastically independent, since they depend on the same stochastic variable $\tilde{r}$. Thus this is not equal to the product of the expectations.

Prove: Invest in risky asset if and only if $E(\tilde{r}) > r_f$

Repeat: $W(a) \equiv E\{U[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}$.

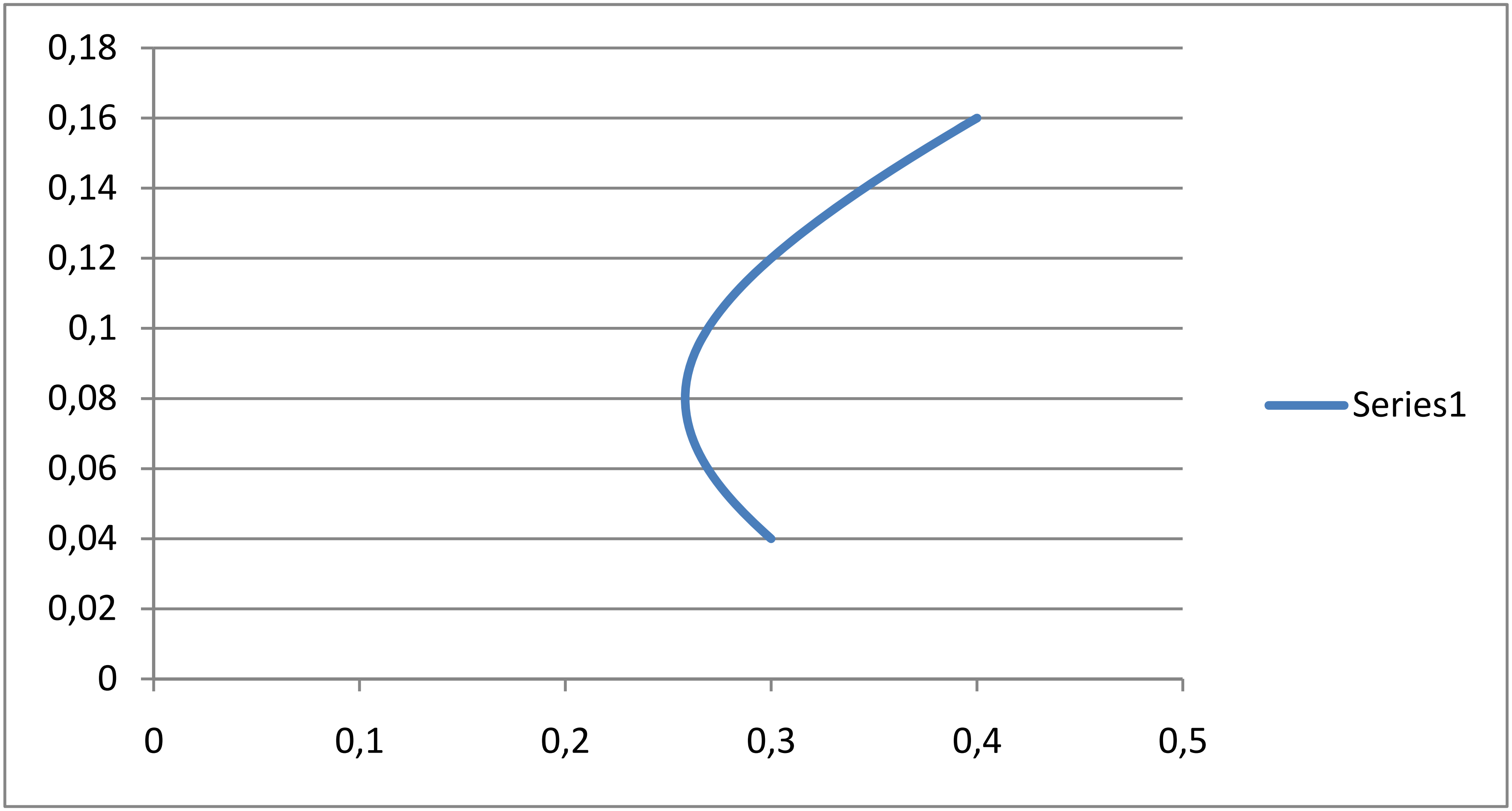
Consider $W''(a) = E\{U''[Y_0(1 + r_f) + a(\tilde{r} - r_f)](\tilde{r} - r_f)^2\}$. The function $W(a)$ will be concave since U is concave. Consider now the first derivative when $a = 0$:

$$W'(0) = E\{U'([Y_0(1+r_f)](\tilde{r}-r_f)\} = U'[Y_0(1+r_f)]E(\tilde{r}-r_f). \quad (3)$$

We find:

- If $E(\tilde{r}) > r_f$, then (3) is positive, which means that E(U)=W will be increased by increasing a from $a = 0$. The optimal a is thus strictly positive.
- If $E(\tilde{r}) < r_f$, then (3) is negative, which means that E(U)=W will be increased by decreasing a from $a = 0$. The optimal a is thus strictly negative.
- If $E(\tilde{r}) = r_f$, then (3) is zero, which means that the f.o.c. is satisfied at $a = 0$. The optimal a is zero.

Of course, $a < 0$ means short-selling the risky asset, which may or may not be possible and legal.



The hyperbola is symmetrical around the horizontal line at $E(r)$=0.08. The merged firm is located at $E(r_z)$=0.14.

Corresponding trees for share and option

$$
S \begin{array}{l} \xrightarrow{\;p^*\;} u \cdot S \\[2ex] \xrightarrow[\;1-p^*\;]{} d \cdot S \end{array}
$$

$$
c \begin{array}{l} \xrightarrow{\;p^*\;} c_u = \max(0, uS - K) \\[2ex] \xrightarrow[\;1-p^*\;]{} c_d = \max(0, dS - K) \end{array}
$$

- Value of call option with expiration one period ahead?
- "Corresponding trees" mean that option value has upper outcome if and only if share value has upper outcome.
- For any K, know the two possible outcomes for c.
- I.e., for a particular option, c_u, c_d known.
- If $K \leq dS$, then $c_d = dS - K, c_u = uS - K$.
- If $dS < K \leq uS$, then $c_d = 0, c_u = uS - K$.
- If $uS < K$, then $c_d = 0, c_u = 0$.
- This third kind of option is obviously worthless.

Replicating portfolios

- Buy a number of shares, Δ, and invest B in bonds.
- Outlay for portfolio today is $S\Delta + B$.
- Tree shows possible values one period later.

$$\begin{tikzcd} & uS\Delta + e^r B \\ S\Delta + B \arrow[ur, dash, "p^*"] \arrow[dr, dash, "1-p^*"'] & \\ & dS\Delta + e^r B \end{tikzcd}$$

- Choose Δ, B so that portfolio *replicates* call.
- "Replicate" (*duplisere*) means mimick, behave like.
- Two equations:

$$\begin{aligned} uS\Delta + e^r B &= c_u, \\ dS\Delta + e^r B &= c_d, \end{aligned}$$

with solutions

$$\Delta = \frac{c_u - c_d}{(u-d)S}, \quad B = \frac{uc_d - dc_u}{(u-d)e^r}.$$

Replicating portfolio, contd.

- (Δ, B) gives same values as option in both states.
- Also called option's *equivalent* portfolio.
- Must have same value now, $c = S\Delta + B$

  $$= \frac{(c_u - c_d)e^r + uc_d - dc_u}{(u-d)e^r} = \frac{(e^r - d)c_u + (u - e^r)c_d}{(u-d)e^r}.$$

  Define $p \equiv (e^r - d)/(u - d)$. (Observe $d \le e^r \le u \Rightarrow 0 \le p \le 1$.) Rewrite formula as

  $$c = \frac{pc_u + (1-p)c_d}{e^r}.$$

- Show $c = S\Delta + B$ by absence-of-arbitrage.
- If observe $c_{\text{obs}} < S\Delta + B$: Buy option, sell pf.
- Cash in $-c_{\text{obs}} + S\Delta + B > 0$ now.
- Keep until expiration.
- In both states, net value is then zero.
- If observe $c_{\text{obs}} > S\Delta + B$: Buy pf., write option.
- Cash in $c_{\text{obs}} - S\Delta - B > 0$ now.
- Keep until expiration.
- In both states, net value is then zero.