Short sales

- Consider investing in a number, X_j, of securities at the price p_{j0} at time zero, with the uncertain price p_{j1} one period ahead.
- Is it possible to hold *negative* quantities, $X_j < 0$?
- Buying a negative number of a security means selling it.
- If you start from nothing, selling requires *borrowing* the security first, then selling, known as a *short sale.*
- Will have to hand it back in period one.
- Will also have to compensate the owner if there has been cash payouts (like dividends) in the meantime.
- Sequence of events:
  - Time 0: Borrow security (e.g. a share of stock) in amount X_j from someone (N.N.)
  - Time 0: Sell security in the market, receive p_{j0}.
  - Between 0 and 1: If payout to security, must compensate N.N. for this.
  - Time 1: Buy back X_j units of security in market.
  - Time 1: Hand it back to N.N.
- Short sale raises cash in period zero, but requires outlay in period one. (Opposite of buying a security.)
- Short-seller interested in falling security prices, $p_{j1} < p_{j0}$.

Overview of today's lecture

- Purpose: Construct *Capital Asset Pricing Model,* CAPM
- Equilibrium model for stock market in closed economy
- First: Describe opportunity set in σ, μ diagram
  - With only two risky assets
  - With many risky assets
  - With one risky and one risk free asset
  - With many risky and one risk free asset
- Then: Optimal choice based on mean-variance preferences
- Then: Consequences for equilibrium prices (more next time)

Mean-var opportunity set, two risky assets

Investor may construct (any) portfolio of (only) two risky assets. What is opportunity set in (σ_p, μ_p) diagram?

$$W_0 = W_{10} + W_{20}$$

$$\tilde{W} = W_{10}(1+\tilde{r}_1)+W_{20}(1+\tilde{r}_2) = W_0\left[\frac{W_{10}}{W_0}(1+\tilde{r}_1) + \frac{W_{20}}{W_0}(1+\tilde{r}_2)\right]$$
$$= W_0[a(1+\tilde{r}_1) + (1-a)(1+\tilde{r}_2)] \equiv W_0(1+\tilde{r}_p).$$

For $j = 1, 2$, let $\mu_j \equiv E(\tilde{r}_j)$, $\sigma_j^2 \equiv \text{var}(\tilde{r}_j)$, and let $\sigma_{12} \equiv \text{cov}(\tilde{r}_1, \tilde{r}_2)$. Then:

$$\mu_p = a\mu_1 + (1-a)\mu_2 \quad \left(\Rightarrow a = \frac{\mu_p - \mu_2}{\mu_1 - \mu_2}\right),$$
$$\sigma_p^2 = a^2\sigma_1^2 + (1-a)^2\sigma_2^2 + 2a(1-a)\sigma_{12}.$$

Taken together:

$$\sigma_p = \sqrt{A\mu_p^2 + B\mu_p + C},$$

where

$$A \equiv \frac{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}{(\mu_1 - \mu_2)^2},$$
$$B \equiv \frac{-2\mu_2\sigma_1^2 - 2\mu_1\sigma_2^2 + 2\sigma_{12}(\mu_1 + \mu_2)}{(\mu_1 - \mu_2)^2},$$
$$C \equiv \frac{\mu_2^2\sigma_1^2 + \mu_1^2\sigma_2^2 - 2\mu_1\mu_2\sigma_{12}}{(\mu_1 - \mu_2)^2}.$$

Opportunity set, two risky assets, contd.

The function $\sigma(\mu) = \sqrt{A\mu^2 + B\mu + C}$ is called an *hyperbola*, the square root of a parabola. Both have minimum points at $\mu = \frac{-B}{2A}$.

Opportunity set, two risky assets, contd.

$$\sigma(\mu) = \sqrt{A\mu^2 + B\mu + C}$$

Asymptotes for hyperbola:

$$\mu \to \infty \Rightarrow \sigma \to \sqrt{A}\mu + \frac{B}{2\sqrt{A}},$$

$$\mu \to -\infty \Rightarrow \sigma \to -\sqrt{A}\mu - \frac{B}{2\sqrt{A}}.$$

Proof (of first part only):

$$\lim_{\mu\to\infty}[\sigma(\mu) - \sqrt{A}\mu] = \lim_{\mu\to\infty} \frac{(\sigma(\mu) - \sqrt{A}\mu)(\sigma(\mu) + \sqrt{A}\mu)}{\sigma(\mu) + \sqrt{A}\mu}$$

$$= \lim_{\mu\to\infty} \frac{(\sigma(\mu))^2 - A\mu^2}{\sigma(\mu) + \sqrt{A}\mu} = \lim_{\mu\to\infty} \frac{B\mu + C}{\sqrt{A\mu^2 + B\mu + C} + \sqrt{A}\mu}$$

$$= \lim_{\mu\to\infty} \frac{B + \frac{C}{\mu}}{\sqrt{A + \frac{B}{\mu} + \frac{C}{\mu^2}} + \sqrt{A}} = \frac{B}{2\sqrt{A}},$$

and the result follows.

Opportunity set, two risky assets, contd.

- When a varies, the hyperbola is traced out.
- $a = 1$ gives the point (σ_1, μ_1).
- $a = 0$ gives the point (σ_2, μ_2).
- Value of a at minimum point, f.o.c.:

$$0 = \frac{d\sigma^2}{da} = 2a\sigma_1^2 - 2(1-a)\sigma_2^2 + (2-4a)\sigma_{12}$$

gives

$$a = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} \equiv a_{\min}.$$

Minimum point of hyperbola. Choose notation $\sigma_1 < \sigma_2$ (so a is share of portfolio in the less risky asset).

Is always $a_{\min} \in [0, 1]$? No, will prove $a_{\min} \in [0, \infty)$.

Proof: Define the *correlation coefficient* $\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$. Then:

$$a_{\min} > 1 \iff \sigma_2^2 - \sigma_{12} > \sigma_1^2 + \sigma_2^2 - 2\sigma_{12}$$

$$\iff \sigma_{12} > \sigma_1^2 \iff \rho_{12} > \sigma_1/\sigma_2,$$

which may or may not be true. (Only general restriction on ρ_{12}, known from statistics theory, is $-1 \leq \rho_{12} \leq 1$.)

Similarly:

$$a_{\min} < 0 \iff \rho_{12} > \frac{\sigma_2}{\sigma_1} > 1,$$

which is impossible.

Hyperbola's dependence on correlation

(D&D, Appendix 6.2)

- Five constants determine shape of hyperbola:

$$\mu_1, \mu_2, \sigma_1, \sigma_2, \sigma_{12}.$$

- Assets' coordinates, (σ_1, μ_1) and (σ_2, μ_2), are not sufficient.
- For fixed values of these four, let σ_{12} vary.
- This is *not* something the investor may choose to do, only a way to illustrate possible situations.
- Easier to discuss in terms of $\rho_{12} \equiv \sigma_{12}/\sigma_1\sigma_2$.
- Consider first what hyperbola looks like for $a \in [0, 1]$.
- Consider first the extremes, $\rho_{12} = \pm 1$.
- For $\rho_{12} = 1$ and $a \in [0, 1]$, find $\sigma_p = a\sigma_1 + (1 - a)\sigma_2$.
- Linear in a, thus also in μ_p.
- In interval between the two points: Straight line.
- Line reaches vertical axis somewhere outside interval. Kink.

Hyperbola's dependence on correlation, contd.

- Opposite extreme, $\rho_{12} = -1$, gives $\sigma_p = \pm[a\sigma_1 - (1-a)\sigma_2]$.
- Also broken line, but now, kink for some $a \in [0, 1]$.
- Specifically, at $a_{\min} = \frac{\sigma_2}{\sigma_1+\sigma_2}$.
- Summing up:
  - For $\rho \in (-1, 1)$, a true (strictly convex) hyperbola.
  - For extreme cases, a broken line.
  - Only for those extreme cases is $\sigma_p = 0$ possible.
- Opportunity set consists of the hyperbola or broken line, only.
- When only two risky assets, impossible to obtain point outside (or "inside") hyperbola.

Mean-var portfolio choice, two risky assets

- Increasing, convex indifference curves.
- Increasing, concave opportunity set (upper half).
- Tangency point will maximize (expected) utility.
- Everyone will choose from upper half of hyperbola.
- Called *efficient set.*
- Choice within efficient set depends on preferences.
- More risk averse: Lower σ.

Mean-var opportunity set: n risky assets, $n > 2$

Let variance-covariance matrix of $(\tilde{r}_1, \ldots, \tilde{r}_n)$ be

$$V = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}.$$

Symmetric, $\sigma_{n1} = \text{cov}(\tilde{r}_n, \tilde{r}_1) = \text{cov}(\tilde{r}_1, \tilde{r}_n) = \sigma_{1n}$.

Diagonal elements were previously called $\sigma_i^2 = \text{var}(\tilde{r}_i)$.

If V has full rank, then impossible to construct portfolio of these assets with $\sigma_p = 0$. (No proof now.) Will concentrate on this case.

(If V has less than full rank, then $\sigma_p = 0$ can be chosen. For $n = 2$, these were the cases $\rho_{12} = \pm 1$.)

Assume now V has full rank. Can be shown: Opportunity set now consists of an hyperbola and the points inside it.

Informally discussed in D&D p. 101, formally on pp. 127–132.

Mean-var opportunity set: n risky assets, contd.

For any μ_p, the agents will want as low σ_p as possible:

$$\min_{w_1,\ldots,w_n} \sigma_p \text{ given } \mu_p$$

This defines the hyperbola called the *frontier portfolio set.*

For any σ_p, the agents will want as high μ_p as possible:

$$\max_{w_1,\ldots,w_n} \mu_p \text{ given } \sigma_p$$

This defines the upper half of the hyperbola. The upper half of the *frontier portfolio set* is known as the *efficient portfolio set.*

Again: *Efficient* means that part of the opportunity set from which the agents will choose, irrespective of their preferences, but within which we cannot predict their choice, since we do not specify their preferences in any more detail.

Mean-var opport. set: One risky, one risk free asset

- Let $\sigma_1 = \sigma_{12} = 0$ in formulae.
- Get linear relation between σ_p and a.
- Thus linear relation between σ_p and μ_p.
- Simplifies. Good reason for working with (σ, μ), not (σ^2, μ).
- Opportunity set broken line.
- Again, upper half is efficient.

Mean-var oppo. set: One risk free, n risky assets, $n > 2$

- Let r_f be rate of return on risk free asset.
- Risk free asset can be combined with any portfolio of risky.
- Everyone will want max μ_p for any given σ_p.
- Assume r_f less than μ at min point.
- (Will return to opposite possibility later.)
- Then: Combination of risk free asset with *tangency portfolio* is efficient. Efficient set is *linear.*

Mean-var portfolio choice: 1 risk free, n risky assets

- Consider now situation with many agents.
- Assume all believe in same means, variances, covariances.
- With mean-variance preferences, all want some combination of risk free asset with *same* portfolio of risky assets, tangency.
- Straight line efficient set.
- Preferences determine preferred location along line.
- Higher risk aversion: Closer to risk free asset.
- Lower risk aversion: Above tangency portfolio: Borrow money (short sell risk free asset) and invest more than W_0 in tangency portfolio.
- "Two-fund spanning": Restriction of opportunity set to r_f and tangency portfolio is just as good as original opportunity set.
- "Separation" of portfolio composition: May leave to a fund manager to make tangency portfolio available.

Equilibrium condition

- Everyone demands same combination of risky assets.
- Necessary condition for equilibrium: This is equal to supply.
- Agent h splits $W_0^h = W_{0f}^h + W_{0M}^h$.
- W_{0f}^h in risk free asset, possibly negative.
- W_{0M}^h in tangency portfolio, strictly positive. (Why?)
- Tangency portfolio has weights $w_{1M}, \ldots, w_{nM}$.
- Per definition $\Sigma\, w_{jM} = 1$.
- Total demand for n risky assets written as vector:

$$\sum_{h=1}^{H} \begin{bmatrix} w_{1M} W_{0M}^h \\ \vdots \\ w_{nM} W_{0M}^h \end{bmatrix} = \begin{bmatrix} w_{1M} \\ \vdots \\ w_{nM} \end{bmatrix} \sum_{h=1}^{H} W_{0M}^h$$

- Total has same value composition as each part.
- This must also be value composition of supply.
- Observable, "market portfolio."
- "Portfolio" here means a vector of weights, summing to one.
- The word "portfolio" may sometimes mean some money amount invested in each of the n assets, a vector not summing to one.

CML, market price of risk

- Everyone combines risk free asset and market portfolio.
- Line through $(0, r_f)$ and (σ_M, μ_M) called *Capital Market Line,* CML,
$$\mu_P = r_f + \frac{\mu_M - r_f}{\sigma_M}\sigma_p.$$
- Slope, $\frac{\mu_M - r_f}{\sigma_M}$, sometimes called *market price of risk.*
- Shows how much must be given up in expected portfolio rate of return in order to reduce standard deviation by one unit.
- All agents have MRS between μ_p and σ_p equal to this.
- Will soon see: This is relevant concept for comparing whole portfolios, but not for individual assets.

Motivating CAPM: Covariances important

- Next derive most important formula in this part of course.
- Model known as the *Capital Asset Pricing Model.* This name also used for the main formula. Formula also called the *Security Market Line* (SML).
- Shows what determines prices of individual assets.
- First motivation: Covariances important.
- Comparing alternative portfolios, when only one of them can be chosen, have assumed *variances* of rates of return are the relevant measure of risk.
- But for individual assets, which can be combined in portfolios, the relevant measure turns out to be a *covariance* with other rates of return.
- Make two simple, motivating arguments first, without reference to any equilibrium model.
- Consider making an *equally weighted* portfolio of n assets, i.e., with all $w_j = 1/n$. Assume that among the rates of return, one has the maximum variance, $\sigma^2_{\max}$. Then

$$\lim_{n\to\infty} \sigma_p^2 = \bar{\sigma}_{ij},$$

the average covariance between rates of return, and

$$\lim_{n\to\infty} \frac{\partial \sigma_p^2}{\partial w_i} = 2\bar{\sigma}_{ij}.$$

Proof of motivating results

Observe that

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}.$$

An equally weighted portfolio has

$$\sigma_p^2 = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} = \frac{1}{n^2} \sum_{i=i}^{n} \sigma_i^2 + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \sigma_{ij}.$$

Observe that the first term satisfies

$$\frac{1}{n^2} \sum_{i=i}^{n} \sigma_i^2 < \frac{1}{n^2} \cdot n \cdot \sigma_{\max}^2 \to 0 \quad \Leftarrow \quad n \to \infty.$$

The second term satisfies

$$\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \sigma_{ij} = \frac{n^2 - n}{n^2} \bar{\sigma}_{ij} \to \bar{\sigma}_{ij} \quad \Leftarrow \quad n \to \infty,$$

which proves the first result. Observe next that for any portfolio,

$$\frac{\partial \sigma_p^2}{\partial w_i} = 2 w_i \sigma_i^2 + 2 \sum_{j \neq i} w_j \sigma_{ij}.$$

Evaluated where all $w_i = 1/n$, this becomes

$$2 \frac{\sigma_i^2}{n} + 2 \frac{n-1}{n} \bar{\sigma}_{ij} \to 2 \bar{\sigma}_{ij} \quad \Leftarrow \quad n \to \infty.$$

Derivation of CAPM formula

- Consider an equilibrium, everyone holds combination of risk free asset and market portfolio.
- Will derive relation between μ_j, σ_j (of any asset, numbered j) and the economy-wide variables r_f, μ_M, σ_M.
- As a *thought experiment* (only), make a portfolio with a fraction a in asset j and a fraction $1-a$ in the market portfolio.
- (Possible, even though M already contains j.)
- For this portfolio p we have

$$\mu_p = a\mu_j + (1-a)\mu_M, \quad \frac{\partial \mu_p}{\partial a} = \mu_j - \mu_M,$$

$$\sigma_p = \sqrt{a^2\sigma_j^2 + (1-a)^2\sigma_M^2 + 2a(1-a)\sigma_{jM}},$$

$$\frac{\partial \sigma_p}{\partial a} = \frac{a\sigma_j^2 - (1-a)\sigma_M^2 + (1-2a)\sigma_{jM}}{\sqrt{a^2\sigma_j^2 + (1-a)^2\sigma_M^2 + 2a(1-a)\sigma_{jM}}},$$

$$\left.\frac{\partial \sigma_p}{\partial a}\right|_{a=0} = \frac{\sigma_{jM} - \sigma_M^2}{\sigma_M}.$$

Illustrating the derivation

(D&D, sect. 7.2, app. 7.1, fig. 7.1)

- Small hyperbola goes through M (i.e., (σ_M, μ_M)).
- At M it has same tangent as large hyperbola: If not, it would have to cross over large hyperbola. But that cannot happen, since large hyperbola is frontier, and j was already available when large hyperbola was formed as frontier.
- The tangent *is* the capital market line.
- Next: Use the equality of these slopes.