The Black-Scholes-Merton formula (Hull 13.5–13.8)

- Assume S_t is a geometric Brownian motion w/drift.
- Want market value at $t = 0$ of call option.
- European call option with expiration at time T.
- Payout at T is $\max(S_T - K, 0)$.
- Assume stock does not pay dividends.
- Three alternative methods lead to same result:
  1. Take limit of binomial model as $n \to \infty, h \to 0$.
  2. Replicating portfolio strategy directly in continuous time.
  3. Find "risk-neutral" expectation of $\max(S_T - K, 0)$.
- Hull (top of p. 292) starts on 2., see exercise 13.17, p. 304.
- Instead does 3. on pp. 307–309.
- Hull p. 256 has reference to article using 1.:
- Cox, Ross, and Rubinstein (1979).
- Today: Will follow Hull; first 2., then 3.
- Result is Black-Scholes-Merton formula,

$$c(S_0, K, T, r, \sigma) \equiv S_0 N(d_1) - K e^{-rT} N(d_2),$$

where N is the standard normal distribution function,

$$d_1 \equiv \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad \text{and} \quad d_2 \equiv d_1 - \sigma\sqrt{T}.$$

Portfolio strategies; replicating vs. risk free

- In binomial model, showed a replicating pf. strategy.
  - Holding Δ shares and B in bonds equals option.
- Hull instead combines share and option to get risk free pf.:
  - Holding Δ shares minus option equals $-B$ bonds.
- In many periods: Need to readjust ...
  - readjust replicating portfolio to replicate option, or
  - readjust risk free portfolio to stay risk free.
- In continuous time: Need to readjust continuously.
- Relies on literal interpretation of "no transaction costs."
- Will show how to determine risk free pf. strategy.
- This pf. strategy must earn risk free interest rate.
- If not: Exists riskless arbitrage opportunity.

Risk free portfolio strategy; share and option

(Hull, pp. 287–288)

- S_t is a geometric Brownian motion with drift, an Itô process,
$$dS = \mu S\ dt + \sigma S\ dz.$$
- Before T: Call option value is function of S_t (or S for short).
- Also function of t (or $T - t$, time until expiration).
- What follows is not limited to a call option, $c(S, t)$.
- Valid for any derivative of S, use notation $f(S, t)$.
- Use Itô's lemma for $f(S, t)$,
$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial f}{\partial S}\sigma S\ dz.$$
- For short intervals Δt:
$$\Delta S = \mu S\ \Delta t + \sigma S\ \Delta z.$$
and
$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) \Delta t + \frac{\partial f}{\partial S}\sigma S\ \Delta z.$$
- Compose portfolio with $\partial f/\partial S$ shares and -1 derivative.
- Value of portfolio is $\Pi = -f + \frac{\partial f}{\partial S} S$.
- Change in value over short Δt is
$$\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t.$$
- This is risk free since there is no Δz.

Differential equation follows from no arbitrage

- The no-arbitrage condition requires $\Delta\Pi = r\Pi\,\Delta t$.
- This implies

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

- This is a partial differential equation (PDE) in $f(S,t)$.
- It has many solutions.
- Only natural, since we have not specified a call option.
- Equation equally valid for put option and other derivatives.
- To obtain a particular derivative, need *boundary condition:*
- Boundary condition for call option is $f = \max(S-K,0)$ when $t = T$.
- Black-Scholes-Merton solves PDE and boundary condition.
- Hull leaves this to the reader, exercise 13.17, p. 304.
- Technical note: Compare B-S-M formula p. 291 and p. 304.
  - The T on p. 291 is replaced by $T-t$ on p. 304.
  - For a particular option, T is fixed; $T-t$ varies over time.
  - Asking how c varies with time means as $T-t$ goes to zero.
  - t increases until it reaches T.
  - t is the time variable relevant for the partial diff. equation.
  - This explains the need for $T-t$ in exercise 13.17.

Option pricing using "risk-neutral" method

- Based on $\hat{S}_t$, an adjusted process for S_t.
- Same starting point point, $\hat{S}_0 = S_0$.
- Same volatility, σ.
- But an expected price increase as if investors were risk neutral.
- $E(\hat{S}_T) = S_0 e^{rT}$ instead of $E(S_T) = S_0 e^{\mu T}$.
- ($E(\hat{S}_T)$ is what Hull calls $\hat{E}(S_T)$.)
- (Also D& D, cf. lecture notes 18 August, p. 6.)
- Market value at time zero is $e^{-rT} E[\max(0, \hat{S}_T - K)]$.
- May split the payoff in two parts:
  - Paying K in case $S_T > K$.
  - Receiving S_T in case $S_T > K$.
- Need expectations for each part.
- Instead of S_T, use $\hat{S}_T$, with probability density $f(\hat{S}_T)$:

$$E(K|\hat{S}_T > K) = \int_K^\infty K f(\hat{S}_T) d\hat{S}_T = K \int_K^\infty f(\hat{S}_T) d\hat{S}_T,$$

  which is equal to $K \Pr(\hat{S}_T > K)$; the other part is

$$E(\hat{S}_T|\hat{S}_T > K) = \int_K^\infty \hat{S}_T f(\hat{S}_T) d\hat{S}_T.$$

- The first is simpler, since K is a constant.

Valuation of obligation to pay K if $S_T > K$

$$\Pr(\hat{S}_T > K) = \Pr(\ln \hat{S}_T - \ln S_0 > \ln K - \ln S_0),$$

where $\ln \hat{S}_T - \ln S_0 \sim \phi((r - \sigma^2/2)T, \sigma^2 T)$, so that $\Pr(\hat{S}_T > K) =$

$$\Pr\left(\frac{\ln \hat{S}_T - \ln S_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} > \frac{\ln K - \ln S_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right),$$

where the variable to the left of the inequality sign is standard normal. This is thus equal to

$$1 - N\left(\frac{\ln K - \ln S_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).$$

The symmetry of the normal distribution means that $1 - N(x) = N(-x)$, so we may rewrite this as

$$N\left(\frac{\ln S_0 - \ln K + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).$$

This means that the valuation of an obligation to pay K if $S_T > K$ is

$$Ke^{-rT}N\left(\frac{\ln S_0 - \ln K + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right),$$

which appears as part of the Black-Scholes-Merton formula.

Valuation of claim to receive S_T if $S_T > K$

Define $h(Q) \equiv \frac{1}{\sqrt{2\pi}} e^{-Q^2/2}$ (a std. normal density), $w \equiv \sigma\sqrt{T}$, $m \equiv \ln S_0 + (r - \sigma^2/2)T$, $Q \equiv (\ln \hat{S}_T - m)/w$.

Then Q is a standard normal variable, and can be rewritten as

$$\frac{\ln \hat{S}_T - \ln S_0 - \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}.$$

From the definition of Q we have $\hat{S}_T = e^{wQ+m}$. The conditional expectation we need is

$$E(\hat{S}_T | \hat{S}_T > K) = E\left(e^{wQ+m} \middle| e^{wQ+m} > K\right) = E\left(e^{wQ+m} \middle| Q > \frac{\ln K - m}{w}\right)$$

$$= \int_{\frac{\ln K - m}{w}}^{\infty} e^{wQ+m} h(Q) dQ.$$

The integrand can be rewritten (Hull, p. 308) as

$$\frac{1}{\sqrt{2\pi}} e^{(-Q^2+2wQ+2m)/2} = e^{m+w^2/2} h(Q - w)$$

The integral can thus be rewritten as

$$e^{m+w^2/2} \int_{\frac{\ln K - m}{w}}^{\infty} h(Q - w) dQ = e^{m+w^2/2} \int_{\frac{\ln K - m}{w}+w}^{\infty} h(Q) dQ =$$

$$e^{m+w^2/2} \int_{\frac{\ln K - m}{w}-w}^{\infty} h(Y) dY,$$

introducing $Y = Q - w$ as a new variable of integration. Clearly, as Q goes from $(\ln K - m)/w$ to ∞, Y goes from $(\ln K - m)/w - w$ to ∞. The integral with Y is the probability that a standard normal variable exceeds $(\ln K - m)/w - w$. Notice that $e^{m+w^2/2} = S_0 e^{rT}$. Also multiply by e^{-rT} to get the valuation of the claim,

$$e^{-rT} E(\hat{S}_T | \hat{S}_T > K) = S_0 N\left(\frac{\ln S_0 - \ln K + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}\right).$$

Conclude: The Black-Scholes-Merton formula

$$c(S_t, K, T-t, r, \sigma) \equiv S_t N(d_1) - Ke^{-r(T-t)} N(d_2),$$

where N is the standard normal distribution function,

$$d_1 \equiv \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad \text{and} \quad d_2 \equiv d_1 - \sigma\sqrt{T-t},$$

(written with S_t and $T-t$ as arguments).

- Together, preceding two pages give the formula.
- Valid for European call options on no-dividend stocks.
- For these, early exercise of American calls is not optimal.
- Thus also valid for American call options on these stocks.
- Or *in periods* when a stock for sure does not pay dividends.
- Can show that the function $c(S_t, K, T-t, r, \sigma)$ is
  - increasing in S_t,
  - decreasing in K,
  - increasing in $T-t$,
  - increasing in r, and
  - increasing in σ,

  cf. the discussion on p. 5 of 27 October.
- Put option values can be found through put-call parity.
- Formula used a lot in practice; also modified, e.g. for dividends.
- Hull's Figs. 9.1–9.2 show properties of formula (also pp 292–293).

Dividends in option pricing

- In section 13.12 Hull considers *known* dividends.
- Both dates and magnitudes are known.
- Much more complicated if one or both are unknown.
- European call: Use $S_t - I$ instead of S_t.
- I is present value of dividends to be paid in (t, T).
- Easily understood from risk-neutral valuation method.
- Call option value is $e^{-r(T-t)} \max(0, S_T - K)$, but S_T must be interpreted as the process of the share value without the dividend, which has a starting value of $S_t - I$ at time t.
- In principle the σ to be used should also reflect this process without dividends (see Hull, fn. 12).
- American call option with dividends: Early exercise?
- 27 Oct. p. 11: If early exercise, then just before div.
- Based on this and *known* dividends (Hull, p. 299):
  - Assume the n dividend dates are $t_1 < t_2 < \ldots < t_n < T$.
  - Corresponding dividends are $D_1, \ldots, D_n$.
  - Consider first whether optimal to exercise at t_n.
  - Hull shows: If $D_n \leq K[1 - e^{-r(T-t_n)}]$, never exercise.
  - If $D_n > K[1 - e^{-r(T-t_n)}]$, exercise if S_{t_n} "big enough."
  - Something similar for earlier dividend dates.
  - No exact formulae.

- Alternative, p. 300: Compare with European options.
- One with T as expiration, another with t_n.
- Use the larger of these two European values as approximation.
- Could maybe extend with more than two dividend dates.

Volatility, σ

- $\sigma = \sqrt{\text{var}[\ln(S_t/S_{t-1})]}$ is called volatility.
- Only variable in Black-Scholes-Merton not directly observable.
- Must be estimated, typically from time-series data.
- If model is true and constant over time, this is easy.
- If time-varying, may use, e.g., last six months.
- (Perhaps also daily, weekly or monthly data make difference.)
- If models of stock price S_t *and* of option value c_t are true:
- Can compare observed option values with theoretical values.
- If assume $c_{\text{obs.},t} = c_{\text{theoretical},t} \equiv c(S_t, K, T-t, r, \sigma)$:
- (And assume for sure no dividends are paid until time T:)
- Only one variable, σ, not directly observable in equation.
- May solve equation for σ, called *implicit volatility.*
- Solution cannot be found explicitly, but by numerical methods.
- Interpretation: Market uses B-S-M; what σ does it believe?
- Forward-looking number, as opposed to time-series, historical.