D&D, sect. 7.4–7.7: CAPM without risk free asset

Main points:

- Consider N risky assets, $N > 2$, no risk free asset. Then the frontier portfolio set is an hyperbola. (Mentioned without proof on p. 11 of 1 September.)
- Can derive version of CAPM without risk free asset. Important if, e.g., there is uncertain inflation.

The version mentioned in the second point is important to understand much of the CAPM literature.

Market portfolio plays important role also in that version of the model, even though it is *not* equal to the risky part of everyone's portfolio.

In lecture today, will relate to parts of the discussion in D & D, and will use their equation numbers. But will simplify, and skip some of their intermediate results which are not necessary for the main results we need.

The new version of the CAPM can be illustrated in the σ, μ diagram we have used previously. Following D & D, the expected rate of return for portfolio p will now be denoted E_p, not μ_p, but this is the same variable.

Observe that Figure 7.6 on p. 134 of D & D is a different diagram. It has σ^2 on the horizontal axis, not σ. The frontier portfolio set is thus a parabola, not an hyperbola, in that diagram. The particular geometrical properties shown there, will not exist in a σ_p, E_p diagram. We will skip Figure 7.6 in the discussion which follows.

Differentiation of vectors and matrices

(See chapter 23 of the math manual by Sydsæter, Strøm and Berck.)

- Derivative of a (scalar) function with respect to an $N \times 1$ vector is the $N \times 1$ vector of derivatives with resp. to each element.
- Derivative of a (scalar) function with respect to an $1 \times N$ vector is the $1 \times N$ vector of derivatives with resp. to each element.
$$x = (x_1, \ldots, x_N) \Longrightarrow \frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N}\right).$$
- Derivative of scalar product: a and x both are $N \times 1$:
$$\frac{\partial}{\partial x}(a^T \cdot x) = a^T. \quad (^T \text{ denotes transpose.})$$
(Generalizes scalar $\partial(b \cdot y)/\partial y = b$.)
- Derivative of $M \times 1$ vector w.r.t. $N \times 1$ vector is $M \times N$ matrix of derivatives.
- Derivative of matrix-vector product: A is $M \times N$, x is $N \times 1$:
$$\frac{\partial}{\partial x}(Ax) = A.$$
- Derivative of quadratic form: A is $N \times N$, x is $N \times 1$:
$$\frac{\partial}{\partial x}(x^T A x) = x^T (A + A^T).$$
(Generalizes scalar $\partial(b \cdot y^2)/\partial y = 2b \cdot y$.)
- Symmetric version of the same: $A = A^T$ is $N \times N$ symmetric:
$$\frac{\partial}{\partial x}(x^T A x) = 2x^T A.$$
- Greek letter iota denotes vector of one's: $\iota = (1, \ldots, 1)^T$.

Frontier portfolio set, no risk free asset

$$\text{Defined by } \min_w \frac{1}{2}\sigma_p^2 \text{ for any expected } E(\tilde{r}_p).$$

- Assume exist N risky assets (securities), no risk free.
- $w_p = \|w_{ip}\|$, the $N \times 1$ vector of portfolio weights.
- Weights must satisfy $w_p^T \iota \equiv \Sigma_{i=1}^N w_{ip} = 1$. (7.10)
- Fundamental data, exogenous in minimization problem, are
  - $e = (E(\tilde{r}_1), \ldots, E(\tilde{r}_N))^T$, $N \times 1$ vector of mean r. of return
  - $V = \|\sigma_{ij}\|$, $N \times N$ cov. matrix of rates of return
- Mean of (r.o.r. of) pf. p is $E_p = w_p^T e = \Sigma_{i=1}^N w_{ip} E(\tilde{r}_i)$. (7.9)
- Variance of (r.o.r. of) portfolio p is $\sigma_p^2 = w_p^T V w_p = \Sigma_{i=1}^N \Sigma_{j=1}^N w_{ip} w_{jp} \sigma_{ij}$.
- Covar. of (r.o.r. of) two pf.s is $\sigma_{p_1 p_2} = w_{p_1}^T V w_{p_2} = \Sigma_{i=1}^N \Sigma_{j=1}^N w_{ip_1} w_{jp_2} \sigma_{ij}$.
- Use matrix notation in solution for frontier portfolio set:
- For any value of E_p: Choose w to obtain minimum σ_p^2.
- Lagrangian $L = \frac{1}{2} w^T V w + \lambda(E_p - w^T e) + \gamma(1 - w^T \iota)$.
- The Lagrangian is a scalar expression, as usual.
- F.o.c.: $\partial L / \partial w = V w - \lambda e - \gamma \iota = 0_N$. (7.8)
- The f.o.c. consists of N scalar equations, here written as an $N \times 1$ vector equation, with an $N \times 1$ vector of zeros on the r.h.s.

Frontier portfolio set, no risk free asset, contd.

Premultiply f.o.c. by V^{-1} to obtain

$$V^{-1}Vw_p - V^{-1}\lambda e - V^{-1}\gamma\iota = 0,$$

where the solution for w is now denoted w_p. This implies

$$w_p = \lambda V^{-1}e + \gamma V^{-1}\iota. \tag{7.11}$$

Using equation (7.9), we find[1]

$$E_p = w_p^T e = e^T w_p = \lambda e^T V^{-1} e + \gamma e^T V^{-1}\iota. \tag{7.13}$$

Combining (7.10) and (7.11), we find

$$1 = w_p^T \iota = \iota^T w_p = \lambda \iota^T V^{-1} e + \gamma \iota^T V^{-1}\iota. \tag{7.14}$$

Equations (7.13) and (7.14) are two scalar equations in the scalar unknowns λ and γ. Thus we can solve for these two in terms of the exogenous e, V. Next we can substitute into (7.11) to find an expression for w_p. This can be written

$$w_p = g + hE_p \tag{7.16}$$

where both g and h are $N \times 1$ vectors determined by the exogenous e, V. We skip the detailed derivations which show that (7.16) is indeed linear, see D & D p. 129.

Equation (7.16) defines the portfolio frontier, i.e., minimizes σ_p for each level of E_p. Will soon show that this is an hyperbola in a σ_p, E_p diagram.

[1] Using the (simple) rule that a scalar product of a column and a row vector is equal to the scalar product of the transposes of the two, put in the opposite order, $x^T y = y^T x$ when x and y are both $N \times 1$.

Interpretation of solution: Frontier is an hyperbola

In equation (7.16), $w_p = g + hE_p$, both g and h are $N \times 1$ vectors, and can be interpreted as portfolios, i.e., vectors of portfolio weights. (7.16) has these implications:

- If we let $E_p = 0$, we find that $w_p = g$ is that vector of portfolio weights which minimizes variance given that the expected rate of return should be zero.
- If we let $E_p = 1$, we find that $w_p = g + h$ is that vector of portfolio weights which minimizes variance given that the expected rate of return should be unity, i.e., 100 percent.
- Portfolio vectors which solve the minimization problem for any other given expected rate of return can be found by taking the appropriate linear combination of these two, g and $g + h$. If you invest 70 percent in g and 30 percent in $g+h$, the resulting expected rate of return is $0.7 \cdot 0 + 0.3 \cdot 1 = 0.3$, i.e., 30 percent. The important insight is that this combination actually gives you the minimum variance among all portfolios with expected rate of return at 30 percent. Why? Because (7.16) tells us that we need weights $w_p = g + 0.3h$ to achieve this, while the weights we actually have are $0.7g + 0.3(g + h)$, which turn out to be the same.
- This means that the portfolio frontier, the collection of all those variance-minimizing portfolios, can be found as various linear combinations of two particular portfolios. Know from earlier that such a curve is an hyperbola in σ, E diagram.

Deriving the zero-beta CAPM

- Will derive a CAPM-like equation without a risk free asset.
- Consider some arbitrary portfolio q and a frontier portfolio p.
- The covariance of the rates of return of these two is

$$\text{cov}(\tilde{r}_p, \tilde{r}_q) = w_p^T V w_q = [\lambda V^{-1} e + \gamma V^{-1} \iota]^T V w_q,$$

  where the expression in square brackets comes from (7.11) above.

- Using rules for transposes of sums and of products, this is

$$= \lambda e^T V^{-1} V w_q + \gamma \iota^T V^{-1} V w_q = \lambda e^T w_q + \gamma = \lambda E(\tilde{r}_q) + \gamma.$$

- Consider now the set of portfolios which have zero covariance with some specific frontier portfolio p. (This is jargon for "portfolios whose rates of return have zero covariance with the rate of return of a frontier portfolio p.") Based on the expression above, these portfolios satisfy

$$E(\tilde{r}_{\text{zero covariance with } p}) = -\gamma/\lambda.$$

- This equation shows that, for some specific frontier portfolio p, those portfolios which have zero covariance with it, all have the same expected rate of return, $-\gamma/\lambda$. They are located on a horizontal line in the σ, E plane.
- Consider in particular that portfolio in this set which has the lowest variance. Denote it as $ZC(p)$, with $E(\tilde{r}_{ZC(p)}) = -\gamma/\lambda$.

Deriving the zero-beta CAPM, contd.

Repeating two equations from the previous page: For some arbitrary portfolio q and some frontier portfolio p, we have

$$\text{cov}(\tilde{r}_p, \tilde{r}_q) = \lambda E(\tilde{r}_q) + \gamma. \tag{7.23}$$

Moreover, for the $ZC(p)$ portfolio relative to p we have

$$E(\tilde{r}_{ZC(p)}) = -\gamma/\lambda, \tag{7.25}$$

or $\gamma = -\lambda E(\tilde{r}_{ZC(p)})$. Plugging this expression for γ into (7.23) gives

$$\text{cov}(\tilde{r}_p, \tilde{r}_q) = \lambda E(\tilde{r}_q) - \lambda E(\tilde{r}_{ZC(p)}). \tag{7.26}$$

Since this is true for any arbitrary portfolio q, it is also true when $q = p$:

$$\text{var}(\tilde{r}_p) = \text{cov}(\tilde{r}_p, \tilde{r}_p) = \lambda E(\tilde{r}_p) - \lambda E(\tilde{r}_{ZC(p)}). \tag{7.27}$$

Combining these two, we have for some arbitrary portfolio q that

$$\frac{E(\tilde{r}_q) - E(\tilde{r}_{ZC(p)})}{\text{cov}(\tilde{r}_p, \tilde{r}_q)} = \lambda = \frac{E(\tilde{r}_p) - E(\tilde{r}_{ZC(p)})}{\text{var}(\tilde{r}_p)},$$

which can be rewritten as

$$E(\tilde{r}_q) = E\left(\tilde{r}_{ZC(p)}\right) + \beta_{pq}\left[E(\tilde{r}_p) - E(\tilde{r}_{ZC(p)})\right]. \tag{7.28}$$

This is very similar to the CAPM equation (the Security Market Line equation), see lecture 8 September 2009, p. 1. But instead of the market portfolio, M, we now have p, which can be any frontier portfolio. And instead of the risk free interest rate, we now have the expected rate of return on the frontier portfolio which has zero covariance with p (which, by the way, of course also holds true for the risk free interest rate).

Deriving the zero-beta CAPM, contd.

- While we have derived the equation we need for the zero-beta CAPM, we still have some way to go.
- One nice feature of the CAPM was that the variables were, at least potentially, observable.
- Would like to express the right-hand side of the equation in terms of observables.
- Will in fact show that the market portfolio is efficient also in this model, i.e., in the absence of a risk free asset.
- First observation: Without a risk free asset, we cannot construct the linear opportunity set known as the Capital Market Line.
- Instead: All agents in the model will choose some mean-variance efficient portfolio on the upper half of the hyperbola known as the frontier portfolio set.
- Will show that when everyone does this, then the market portfolio is also on that same upper half, i.e., the market portfolio is itself an efficient portfolio.
- The reason is that the market portfolio is a convex combination of the portfolios of all the agents. (A convex combination is a linear combination in which all coefficients are between zero and unity.)
- Why is this so?

Market portfolio as convex combination

Agent k has wealth W_0^k and invests in a portfolio vector

$$w^k = \begin{pmatrix} w_1^k \\ \vdots \\ w_N^k \end{pmatrix},$$

which gives money amounts

$$\begin{pmatrix} w_1^k \\ \vdots \\ w_N^k \end{pmatrix} \cdot W_0^k = \begin{pmatrix} w_1^k \cdot W_0^k \\ \vdots \\ w_N^k \cdot W_0^k \end{pmatrix}$$

invested in the N risky assets. The market portfolio in money amounts is then

$$\begin{pmatrix} w_1^1 \\ \vdots \\ w_N^1 \end{pmatrix} \cdot W_0^1 + \ldots + \begin{pmatrix} w_1^K \\ \vdots \\ w_N^K \end{pmatrix} \cdot W_0^K.$$

If we divide this by the total wealth of all K agents, $W_0 = \Sigma_{k=1}^K W_0^k$, we find the market portfolio expressed as relative weights,

$$\begin{pmatrix} w_1^1 \\ \vdots \\ w_N^1 \end{pmatrix} \cdot \frac{W_0^1}{W_0} + \ldots + \begin{pmatrix} w_1^K \\ \vdots \\ w_N^K \end{pmatrix} \cdot \frac{W_0^K}{W_0}.$$

This is, indeed, a convex combination of the individual portfolio vectors, with weights

$$\frac{W_0^1}{W_0}, \ldots, \frac{W_0^K}{W_0},$$

all between zero and unity.

The zero-beta CAPM

Since the market portfolio is efficient in this model, we can introduce it instead of p in equation (7.28), and find

$$E(\tilde{r}_q) = E\left(\tilde{r}_{ZC(M)}\right) + \beta_{Mq}\left[E(\tilde{r}_M) - E(\tilde{r}_{ZC(M)})\right]. \qquad (7.29)$$

Since this holds for any arbitrary portfolio q, it also holds for any individual asset j:

$$E(\tilde{r}_j) = E\left(\tilde{r}_{ZC(M)}\right) + \beta_{Mj}\left[E(\tilde{r}_M) - E(\tilde{r}_{ZC(M)})\right]. \qquad (7.30)$$

This is the zero-beta CAPM. The name is due to the fact that the risk free interest rate is replaced by the expected rate of return on a portfolio which has a beta of zero, relative to the market portfolio. Even though this portfolio cannot be easily observed, equation (7.30) has testable implications.

Testing the CAPM is not covered in this course. Just to give one clue: If (7.30) is thought of as a regression, we now have the problem that we are missing data for $E(\tilde{r}_{ZC(M)})$, as opposed to r_f in the standard CAPM. But the fact that $\tilde{r}_{ZC(M)}$ is uncorrelated with $\tilde{r}_M$ means that we can still get consistent estimates of β.

Two differences between the zero-beta CAPM and the standard CAPM (with a risk free asset), apart from the difference in the equations:

- Every agent does not hold the same combination of risky assets.
- The MRS between σ_p and E_p (previously called μ_p) is not the same for all agents.