

Instructions:

For all students: Please try to solve these exercises before the seminars. It is generally a better idea to work a little with all questions than to work a lot with only a few of them.

Except for the first seminar, I will ask for volunteers to prepare suggested solutions to the exercises. This has two purposes: Those who prepare something, will learn from it. Also, the others will learn, both from correct and mistaken parts of the suggestions. The organization of this, including distribution of the suggested solutions, will be discussed in the first seminar.

Complete, correct solutions to all problems will *not* be made available (except if/when the volunteers' contributions can be regarded as such). You will have to take part in the seminars in order to get all the information, corrections, discussions, modifications, etc. However, if there are some parts which we do not get through because of limited time, solutions to those remaining parts (or extensive comments to the volunteers' suggested solutions) will be made available.

In the texts of the exercises, the notation is sometimes inconsistent. For instance, the risk free interest rate can be r_f in one exercise, but R_f in another. In some cases there is no indication of which variables are stochastic, i.e., the risky rates of return are written as r_j , not \tilde{r}_j . This happens because the problems are collected from various sources, and you just have to get used to this. But within each problem, there should be consistency.

Exercises for seminar no. 1, 7 Sept.

Consider the function $U(C) \equiv ae^{bC} + d$, where a , b , and d are constants, and $e \approx 2.718$ is the well-known constant. C is consumption.

(1)

1(a)

What condition(s) must be satisfied by a , b , and d in order for $E[U(C)]$ to properly represent the preferences of a risk averse person who maximizes von Neumann-Morgenstern expected utility? What are the coefficients of absolute and relative risk aversion, $R_A(C)$ and $R_R(C)$, for this U function?

Consider an individual planning for a future period, with the given U function with that/those conditions you stated in part 1(a). The individual has wealth Y_0 to be divided between two financial investments, Y_f (risk free) and Y_r (risky). The future consumption is equal to the sum of the future values of these investments. Y_f will increase by the factor $R_f \equiv 1 + r_f$, where r_f is the risk free interest rate. Y_r will increase similarly by the factor $\tilde{R} \equiv 1 + \tilde{r}$, where \tilde{r} is a risky rate of return (— this may in fact be a decrease for low

outcomes of \tilde{r}). The individual regards Y_0 , R_f , and the probability distribution of \tilde{R} as exogenously given.

In what follows, you should discuss both the case in which short selling is allowed and the case in which it is not.

1(b)

Describe the individual's maximization problem and its solution.

1(c)

Discuss the statement "Optimal Y_r does not depend on the size of Y_0 ."

Assume in the following that \tilde{R} has only two possible outcomes, R_1 and R_2 , and that $R_1 > R_2$. The probability of R_1 is p .

1(d)

Describe the solution to the maximization problem in this case. Show that under some conditions the optimal Y_r can be written as

$$Y_r^* = \frac{\ln \frac{p(R_1 - R_f)}{(1-p)(R_f - R_2)}}{-b(R_1 - R_2)}.$$

1(e)

Find out whether Y_r^* as given by the formula above is increased or decreased by changes in p , R_f , and b . Try to give intuitive explanations for these effects.

1(f)

Could Y_r^* from the formula exceed Y_0 ? What would the individual do then if borrowing is not allowed?

1(g)

Could Y_r^* from the formula be negative? What would the individual do then if short selling is not allowed?

1(h)

Maintain the assumption that $R_2 < R_1$. What will happen if $R_1 < R_f$? What will happen if $R_2 > R_f$? Can these situations occur?

1(i)

What would be the solution for parts 1(a) – 1(d) if the individual is instead attracted to risk?

(2)

(Exam ECON4515 Spring 2006, problem 2, slightly rewritten and expanded)

There are two periods: period zero (today) and period 1 (next period). In the next period there are three possible states of the world: recession (with probability 0.3), normal growth (with probability 0.4), and expansion (with probability 0.3). There are two stocks (stock A and stock B) in the economy. The stocks will yield payoffs next period according to the following table:

Stock	Recession	Normal Growth	Expansion
A	6	12	14
B	15	5	10

- (a) Can you rank the stocks by the criterion of first order stochastic dominance? If you can rank them, which stock first order stochastically dominates the other? You may use a graph to answer this question if you choose to.
- (b) Can you rank the stocks by the criterion of second order stochastic dominance? If you can rank them, which stock second order stochastically dominates the other? You may use a graph to answer this question if you choose to.

Consider an investor with the following utility function:

$$u(w) = \sqrt{w}.$$

After spending what is needed for consumption today, the investor has a current wealth of 10 plus one share of stock A. The investor can either keep the share until next period or sell it now. Apart from the share, savings will earn a risk free interest rate of zero (for simplicity).

- (c) What is the minimum price the investor would be willing to sell the stock for? Is this more or less than the expected payoff from this share of stock? Why?

Assume that the current wealth is 10000 plus one share of stock A, instead of 10 plus one share of stock A.

- (d) What is the minimum price the investor would be willing to sell the stock for? How do you explain the difference between this answer and your answer under part 2(c)?

Exercises for seminar no. 2, 21 Sept.

(1)

(Part of exam question for ECON460, fall 2002, slightly rewritten.)

In this exercise you are asked to sketch two different opportunity sets in the same diagram, and then to consider what can be said about what choices an investor would make.

Consider a pair of risky securities, 1 and 2, with rates of return $r_j, j = 1, 2$, with the properties $\mu_1 = E(r_1) = 0.16$, $\sigma_1^2 = \text{var}(r_1) = 0.2^2 = 0.04$, $\mu_2 = E(r_2) = 0.08$, and $\sigma_2^2 = \text{var}(r_2) = 0.3^2 = 0.09$.

Two additional pieces of information are needed in order to define the opportunity set of portfolios from the two securities: Short selling may or may not be allowed (see below). With regard to the covariance, you are asked to consider two cases:

Case (i): The covariance between the rates of return is $\sigma_{12} = \text{cov}(r_1, r_2) = 0.045$.

Case (ii): The covariance between the rates of return is $\sigma_{12} = \text{cov}(r_1, r_2) = 0.015$.

1(a)

For case (i): Determine the composition of that portfolio of the two securities which has the minimum variance, and then show that this variance is $\sigma_{mv}^2 = 0.63/16 \approx 0.198^2$. Do likewise for case (ii), with the resulting $\sigma_{mv}^2 = 0.54/16 \approx 0.184^2$.

1(b)

Sketch the two hyperbolae in the same (σ, μ) diagram. Remember that each hyperbola is symmetric around a horizontal line at the μ level which gives the minimum variance (and minimum standard deviation).

1(c)

What are the two opportunity sets (for cases (i) and (ii), resp.) if short selling is allowed? If you only know that an investor is risk averse with mean-variance preferences, can you determine whether the investor would prefer to have the opportunity set given by case (i) or that given by case (ii)?

1(d)

What are the two opportunity sets if short selling is *not* allowed? If you only know that an investor is risk averse with mean-variance preferences, can you determine whether the investor would prefer to have the opportunity set given by case (i) or that given by case (ii)?

(2)

The lecture notes for 1 September describe short selling of a risky security on p. 1. Discuss whether this description is relevant (in practice, not only in theory) for two other forms of investment: (i) A risk free security. (ii) A risky real investment project.

(3)

The lecture notes for 1 September consider situations with one or more risky assets, but there is never more than one risk free asset. Explain why a situation with many different risk free assets is not an interesting situation to consider.

(4)

In the lecture notes for 1 September it was implicitly assumed that the risky assets have different expected rates of return. What can you say about the opportunity set in a situation with only two risky assets when these have the same expected rate of return?

(5)

In the lecture on 1 September it was claimed that the frontier portfolio set with n risky assets, solving

$$\min_{w_1, \dots, w_n} \sigma_p \text{ given } \mu_p$$

is a hyperbola, but there was no proof of this. For the subsequent derivation of the CAPM, however, there was no use for a formula describing the graph of the frontier portfolio set.

5(a)

Explain why the derivation must be based on an assumption (or even better, a proof) that the graph of $\sigma(\mu)$ (the frontier portfolio set transposed with μ as the argument of the function) is convex.

5(b)

Explain why we can be sure that the graph of $\sigma(\mu)$ is indeed convex. (Hint: Assume the opposite, that the curve is concave for some segment, say between μ_1 and μ_2 . Why is that at odds with what you know about combining two risky assets?)

Exercises for seminar no. 3, 12 Oct.

(1)

(Exam Spring 2008, problem 1, slightly rewritten.)

Consider a limited-liability company with shares traded in the stock market. Assume there are no taxes and that the company does not need to borrow. The company is considering to undertake its first real investment project.

It must make a choice in period zero of the output quantity to produce, $Q \geq 0$, which will require a quantity $C(Q)$ of an input factor. There are no other costs. $C(Q)$ is an increasing and convex function. Both quantities, Q and $C(Q)$, will be known with full certainty as soon as the decision on Q has been made, and there is no way to change these later. The output will be ready for sale in period one at an output price \tilde{P} , which is uncertain as seen from period zero.

Two alternative cases for payment for the input factor are considered in parts 1(a) and 1(b) below.

1(a)

Assume that the input factor must be paid for in period zero at a known price W_0 per unit, so that the cost of the input will be $W_0C(Q)$.

Determine the optimal choice of Q based on the Capital Asset Pricing Model. Can you be sure that the company will choose to produce at all? Are the second-order condition for a maximum satisfied?

How does the optimal Q depend on

- i. The risk aversion of the owners of the company?
- ii. The riskiness of the output price?

Give a verbal interpretation of your answers.

1(b)

Assume instead that the input factor must be paid for in period one at a price \tilde{W} per unit. This factor price is uncertain as seen from period zero. Maintain the other assumptions from part 1(a). Again determine the optimal choice of Q .

How does the optimal Q depend on

- i. The risk aversion of the owners of the company?
- ii. The riskiness of the output price?
- iii. The riskiness of the factor price?

Give a verbal interpretation of your answers.

1(c)

Assume that in period zero, after the company has announced its plans to start the project, the value of the shares of the company will be equal to the market's valuation of the company's net cash flow from the project in period one.

Under the assumptions in part 1(a): What will be the relationship between the beta of the shares and the similar risk measure for the output price, the beta of $\tilde{P}/V(\tilde{P})$?

Under the assumptions in part 1(b): What will be the relationship between the beta of the shares and the similar risk measures for the output price and the factor price? How does this differ from the preceding answer (the beta from 1(a)) if the factor price has no systematic risk? Interpret the answer. How does it differ if the factor price has the same beta as the output price?

(2)

2(a)

Consider the standard version of the CAPM, illustrated in a (σ, μ) diagram. Assume that the “risk free asset” is really represented by a bank, which offers deposits and loans at the same interest rate r_f . Illustrate in the diagram what kind of preferences will induce a person to borrow from the bank, and what kind of preferences will induce deposits.

2(b)

Assume instead that the bank offers risk free deposits with an interest rate of r_d and risk free loans with an interest rate of r_ℓ , which exceeds r_d . Show that in this situation the investors will divide themselves into three different groups, depositors, borrowers, and some who prefer not to use the bank.

(3)

Consider an economy in which the Capital Asset Pricing Model holds. The economy has only two types of shares with risky future values. Share i has a rate of return of \tilde{R}_i , for $i = 1, 2$. In addition the agents may save or borrow at a risk free interest rate R_f . It is not necessary to derive the model results.

3(a) Find the market portfolio’s expected rate of return, and the variance of this rate of return, when the following information is given:

$$E(\tilde{R}_1) = 0.1, \quad E(\tilde{R}_2) = 0.2, \quad \text{var}(\tilde{R}_1) = 0.06^2 = 0.0036,$$

$$\text{var}(\tilde{R}_2) = 0.18^2 = 0.0324, \quad \text{cov}(\tilde{R}_1, \tilde{R}_2) = 0.002,$$

the total market value of all shares of type 1 is equal to the total market value of all shares of type 2.

3(b) Show that the given information is sufficient to determine the magnitude of R_f , and try to find this magnitude.

3(c) Find the minimum-variance portfolio within the set of portfolios composed from the two shares. Determine the composition of this portfolio, its expected rate of return, and the variance of this rate of return.

Exercises for seminar no. 4, 26 Oct.

(1)

Consider an economy with only two individuals and one future period, in which only two possible states of the world may be realized. The two individuals both have von Neumann-Morgenstern preferences and are risk averse. They both agree that the states have probabilities π_1 and $\pi_2 = 1 - \pi_1$. Individual i will have an exogenously given income Y_{is} in state s , but may instead obtain consumption c_{is} in that state by buying and selling state-contingent claims in a market. Only the two individuals take part in the trade, but we nevertheless suppose that a competitive equilibrium is obtained. Each individual's budget for buying the claims consists solely of the equilibrium values of that individual's incomes, Y_{is} , i.e., \hat{Y}_i is the budget of individual i , $\hat{Y}_i \equiv p_1 Y_{i1} + p_2 Y_{i2}$, where p_s is the price of a claim to one unit income in state s .

1(a)

Write down the optimization problem for individual i , and find the first-order conditions for a maximum, without specifying the shape of the utility function. Why can we assume that the second-order conditions are satisfied?

1(b)

Assume both individuals have a utility function of the form $U_i(c_{is}) \equiv -e^{-\alpha_i c_{is}}$, where α_i is an individual-specific, positive constant. Show that under the assumption that there is an interior solution, the optimal c_{i1} can be written as

$$c_{i1} = \frac{\hat{Y}_i - \frac{p_2}{\alpha_i} \ln\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)}{p_1 + p_2}$$

1(c)

Define $\alpha \equiv 1/(1/\alpha_1 + 1/\alpha_2)$, and define Y_s (with only one subscript, without hat) as $Y_s \equiv Y_{1s} + Y_{2s}$. Show that in equilibrium the relative price is given as

$$p \equiv \frac{p_1}{p_2} = \frac{\pi_1}{\pi_2} e^{\alpha(Y_2 - Y_1)}$$

1(d)

Give an economic interpretation of how the equilibrium relative price depends on π_1/π_2 , on $Y_2 - Y_1$, and on α_1 and α_2 .

1(e)

Assume now $Y_1 = Y_2$. What is now the formulae for the relative price and the optimal consumptions? Give an economic interpretation of this case.

1(f)

Assume now that in addition to the future period discussed above, there is a period we may call “today.” Consider two investment projects which require the same outlay today. The first gives an income x in the future period, the same irrespective of which state is realized. The second gives an income z in state 1, but nothing in state 2. What do the equilibrium prices from (c) above tell us about (i) the agents’ rankings of these two projects, and (ii) the agents’ willingness today to pay for the projects (i.e., what are the maximum outlays today that would lead the agents to accept the projects)?

(2)

Question 5.24. in Hull, p. 125.

Exercises for seminar no. 5, 9 Nov.

(1)

- 1(a) Explain the concept “risk free arbitrage opportunity.” Explain briefly how this concept is used to prove relationships between the market values of various securities. Show that under some assumptions, the interest rates on two risk free bonds must be the same.
- 1(b) Assume that the binomial share and option model is valid: Consider a share which for sure does not pay any dividend in the periods we focus on. All agents know that if the share price at time t is S , then the share prices at $t + 1$ will be uS or dS . Consider a European call option with two periods left to maturity, when the share price today is $S = 10.00$, the exercise price of the option is $K = 13.19$, the one-period interest rate factor is $e^r = 1.1$, the factor u is 1.2, and the factor d is 1.0. Assume that a price of 0.50 is observed for this option. Show that this creates a risk free arbitrage opportunity, and show how to take advantage of this during the time to the maturity date of the option.

(2)

2(a) Consider a capital market with risk averse agents, in which the standard assumptions of the binomial option value model hold. Assume $e^r = 1.25$, and consider a share with $S_0 = 1.0$, while $\Pr(S_1 = 1.3) = 0.9$ and $\Pr(S_1 = 1.0) = 0.1$. It is certain that the share pays no dividends. A European call option on this share, with $K = 1.15$ and expiration at $t = 1$, sells at $t = 0$ at the price 0.05. How can this arbitrage opportunity be used? What should be sold and bought? What is the arbitrage profit at $t = 0$? What piece of information above is unnecessary?

2(b) Another option is like the first one, but with a different exercise price. Its equilibrium value at $t = 0$ is 0.28. Find the exercise price! (This is a bit tricky.)

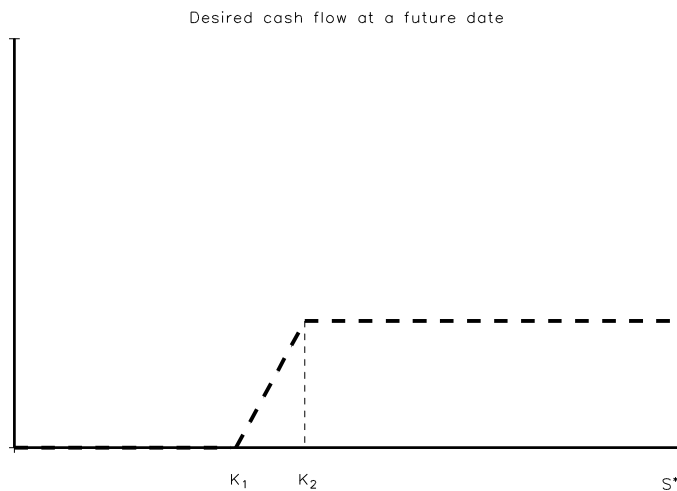
(3)

Question 11.17. in Hull, p. 258.

Exercises for seminar no. 6, 20 Nov.

(1)

Assume that the Black-Scholes formula holds, and that an investor wants a cash flow one year from today of the following form:



The dashed, broken line shows the cash flow as a function of the price of a particular share one year from today. The angle is 45 degrees. (The line extends to infinity at the level 0.31.)

1(a) Show that the cash flow can be obtained (exactly) by using two call options with exercise prices $K_1 = 1$ and $K_2 = 1.31$, respectively, to form a portfolio.

1(b) What is the price of the portfolio when the interest rate factor is $e^r = 1.09417$, and the share has $S = 1$ (today) and $\sigma = 0.2$? (In order to find the answer, you need a table of the cumulative normal distribution function, N , or a computer program which calculates N .)

(2)

Consider a real investment project lasting for one period only, with deterministic costs, $B = 1200$, producing a deterministic quantity, $A = 100$, of some commodity. The output price is P_t , which has uncertain future values. Revenue and costs of the project appear within the same time period.

At $t = 0$ there are traded forward contracts on the commodity for delivery at $t = 1$. These have a forward price, $F_{01} = 19$. Today's price is $P_0 = 18$. The output price follows a GBM with $\sigma = 0.29$. The risk free interest rate factor is $e^r = 1.125$.

Find the value of having a claim to the project opportunity at $t = 0$ under three alternative assumptions:

2(a) The project is undertaken at $t = 0$.

2(b) One commits at $t = 0$ to undertaking the project at $t = 1$.

2(c) After P_1 is known, one will have the choice of undertaking the project at $t = 1$ or never.

You will need numbers like $N(1.73) = 0.9582$ and $N(1.44) = 0.9251$.

(3)

Questions 12.15. and 12.16. in Hull, p. 274.

(4)

Question 13.29. in Hull, p. 306.