Final Exam ECON4510 «Finance Theory» Spring 2024

Solution: Answers will follow. For instructional purposes, these may be more elaborate than what was expected from the students writing the final exam.

(10 points) 1. *Investing in bonds*

Suppose you purchase a 30-year, zero-coupon bond with a yield to maturity of 6%. You hold the bond for five years before selling it. (Hint: first compute the price at year 0, then at year 5.)

(a) If the bond's yield to maturity is 6% when you sell it, what is the internal rate of return of your investment?

(b) If the bond's yield to maturity is 7% when you sell it, what is the internal rate of return of your investment?

(c) If the bond's yield to maturity is 5% when you sell it, what is the internal rate of return of your investment?

(d) Even if the bond has no chance of default, is your investment risk free if you plan to sell it before it matures? Explain.

Solution: For ease of exposition, but without any loss of generality, we assume here that the face value of the bond is 100. Obviously, we could have assumed that the face value was any number. We will formally show that at the end of this secion.

For all sub-questions,

$$\text{purchase price} = \frac{100}{1.06^{30}} = 17.41$$

(a)

$$\text{Sales price} = \frac{100}{1.06^{25}} = 23.30$$

$$\text{Return} = \left(\frac{23.30}{17.41}\right)^{\frac{1}{5}} - 1 = 6.00\%$$

Since YTM is the same at purchase and sale, IRR is the same as the initial YTM.

(b)

$$\text{Sales price} = \frac{100}{1.07^{25}} = 18.42$$
$$\text{Return} = \left(\frac{18.42}{17.41}\right)^{\frac{1}{5}} - 1 = 1.13\%$$

Since YTM rises, IRR is less than the initial YTM.

(c)

$$\text{Sales price} = \frac{100}{1.05^{25}} = 29.53$$
$$\text{Return} = \left(\frac{29.53}{17.41}\right)^{\frac{1}{5}} - 1 = 11.15\%$$

Since YTM falls, IRR is greater than the initial YTM.

(d) If you sell prior to maturity, you are always, in addition to default risk, exposed to the risk that the yield to maturity may change

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For general face value x:

$$\text{purchase price} = \frac{x}{1.06^{30}}$$

(a)

$$\text{Sales price} = \frac{x}{1.06^{25}}$$
$$\text{Return} = \left(\frac{\frac{x}{1.06^{25}}}{\frac{x}{1.06^{30}}}\right)^{\frac{1}{5}} - 1 = \left(\frac{1.06^{30}}{1.06^{25}}\right)^{\frac{1}{5}} - 1 = 6.00\%$$

Since YTM is the same at purchase and sale, IRR is the same as the initial YTM.

(b)

$$\text{Sales price} = \frac{x}{1.07^{25}}$$
$$\text{Return} = \left(\frac{\frac{x}{1.07^{25}}}{\frac{x}{1.06^{30}}}\right)^{\frac{1}{5}} - 1 = \left(\frac{1.06^{30}}{1.07^{25}}\right)^{\frac{1}{5}} - 1 = 1.13\%$$

Since YTM rises, IRR is less than the initial YTM.

(c)

$$\text{Sales price} = \frac{x}{1.05^{25}}$$

$$\text{Return} = \left(\frac{\frac{x}{1.05^{25}}}{\frac{x}{1.06^{30}}}\right)^{\frac{1}{5}} - 1 = \left(\frac{1.06^{30}}{1.05^{25}}\right)^{\frac{1}{5}} - 1 = 11.15\%$$

Since YTM falls, IRR is greater than the initial YTM.

(d) And again, you sell prior to maturity, you are always, in addition to default risk, exposed to the risk that the yield to maturity may change

(10 points) 2. *The CAPM*

(a) Widget shares have a beta of 1.5. The expected risk-free rate is 3%, and the expected returns on the market portfolio are 8%. In equilibrium, supposing that the CAPM holds, what should be the expected return on Widget shares?

(b) Superwidget shares have a beta of 1.2. You invest half of your money in Widget and half of your money in Superwidget. What are your expected returns?

(c) You buy a portfolio of 5 risk-free bonds with a price of NOK 950 each and 10 shares in Widget with a current price of NOK 120. What is the expected return on the portfolio?

(d) What is the beta of the portfolio?

Solution:

(a)

$$\text{Expected return} = 0.03 + 1.5 \cdot (0.08 - 0.03) = 10.5\%$$

(b) The beta of the portfolio is

$$0.5 \cdot 1.5 + 0.5 \cdot 1.2 = 1.35$$

and the expected return is

$$0.03 + 1.35 \cdot (0.08 - 0.03) = 9.75\%$$

(c) The expected return is total investment is

$$\frac{5 \cdot 950 \cdot 0.03 + 10 \cdot 120 \cdot 0.105}{5 \cdot 950 + 10 \cdot 120} = 4.5\%$$

(d) The beta of the total investment is

$$\frac{5 \cdot 950 \cdot 0 + 10 \cdot 120 \cdot 1.5}{5 \cdot 950 + 10 \cdot 120} = 0.30$$

Double-check: expected return with beta 0.30

$$0.03 + 0.3 \cdot (0.08 - 0.03) = 0.03 + 0.015 = 4.5\%$$

(10 points) 3. *The lognormal model and aggregate puzzles*

Assume utility is additively separable CRRA and consumption growth is lognormally distributed such that the pricing kernel is

$$m = \delta \left(\frac{c_1}{c_0}\right)^{-\alpha} \quad \text{where} \quad \frac{c_1(z)}{c_0} = e^z, \quad \text{with} \quad z \sim \mathcal{N}(\kappa_1, \kappa_2)$$

and equity levered claim on consumption; dividend growth is a function of consumption growth for some arbitrary value of λ

$$d^e(z) = \left[\frac{c_1(z)}{c_0}\right]^\lambda$$

then

$$R_f = \frac{1}{\delta} e^{\alpha\kappa_1 - \alpha^2\kappa_2/2} \quad \text{and} \quad \frac{\mathbf{E}[R_m]}{R_f} = e^{\lambda\alpha\kappa_2}$$

so

$$\log R_f = -\log\delta + \alpha\kappa_1 - \frac{1}{2}\alpha^2\kappa_2 \quad \text{and} \quad \log \mathbf{E}[R_m] - \log R_f = \lambda\alpha\kappa_2$$

Using the two equations on the last line, please *briefly* explain the aggregate asset pricing puzzles and how the model with additively separable utility and lognormal consumption growth cannot quantitatively account for asset pricing movements, and the tensions between the "equity premium puzzle" and "risk-free rate puzzle".

Solution: Sufficient: to quantitatively account for the historically observed equity premium, α, the risk-aversion coefficient, must be high and much, much higher than we otherwise use in applied micro and macro. That is a problem by itself, but, in addition, as these equations show if the risk-aversion coefficient is high, then the risk-free rate is much higher than empirically observed.

More elaborate for later reference, we can illustrate with some numbers: long-run aggregate consumption growth (κ_1) is about 1.5 percent and its standard deviation ($\sqrt{\kappa_2}$) is about 3 percentage points. If the time preference rate (δ) is 0.99, and the risk-aversion coefficient (α) is equal to 1 we have

$$\begin{aligned}\log R_f &= -\log\delta + \alpha\kappa_1 - \frac{1}{2}\alpha^2\kappa_2 \\ &= 0.01 + 1\cdot 0.015 - \frac{1}{2}\cdot 1^2 \cdot 0.0009 = 2.5\%\end{aligned}$$

and if $\alpha = 2$, $\log R_f = 4\%$.

With $\alpha = 1$ and $\lambda = 3$, which is a very high estimate for λ, we have

$$\log \mathbf{E}[R_m] - \log R_f = 3\cdot 1\cdot 0.0009 = 0.0027 = 0.27\%$$

which is much lower than the observed equity risk premium.

Alternatively, we could solve the second equation for the value of the risk aversion parameter that would give us an equity risk premium of 5 percent, which is $\alpha = 19$. But with $\alpha = 19$ (and still $\delta = 1$) the risk free rate is

$$\log R_f = 0.01 + 19 \cdot 0.015 - \frac{1}{2} \cdot 19^2 \cdot 0.0009 = 13.25\%$$

which is much higher than observed risk-free real rates.

(20 points) 4. *From Gordon growth to valuation multiples*

In this question, we will investigate how to translate the value from the Gordon growth model into multiples, including EV/EBITDA, P/E, and EV/Sales.

One of the simplest valuation techniques is the Gordon growth model:

$$\text{Enterprise value} = \frac{\text{Free cash flow}}{\text{Cost of capital} \ - \ \text{growth}}$$

Enterprise value (EV) of the firm equals the equity market capitalization plus debt and other liabilities, minus cash.

Free cash flow (FCF) is the cash a firm generates that is free to be distributed to the holders of debt and equity. Formally, it equals net operating profit after taxes (NOPAT) minus investments in future growth. These investments include changes in working capital and capital expenditures.

The cost of capital is the weighted average cost of capital (WACC) for companies financed with both debt and equity. Growth captures the expected increase in FCF over time.

In class, we decomposed free cash flow (FCF) as follows:

$$\text{FCF} = \text{EBIT}(1-t) - (\text{capital expenditures} - \text{depreciation}) - \Delta \cdot \text{working capital}$$

where EBIT stands for earnings before interest and taxes and t stands for tax rate. The first term on the right side of the equation is net operating profit after taxes (NOPAT) and the second two terms are investments.

(a) When

$$\text{EBITDA} = \text{EBIT} + \text{depreciation}$$

and

$$\text{Cost of capital} = \text{WACC}$$

show that the multiple enterprise value (EV) over EBITDA can be expressed as

$$\frac{\text{EV}}{\text{EBITDA}} = \frac{\text{Depreciation} \cdot t/\text{EBITDA}}{\text{WACC} \ - g} + \frac{1-t}{\text{WACC} \ - g} - \frac{\text{capital expenditures/EBITDA}}{\text{WACC} \ - g} - \frac{\Delta \text{ in working capital/EBITDA}}{\text{WACC} \ - g}$$

Assume we have the following numbers:

Sales	500
EBIT	100
Depreciation	25
EBITDA	125
Tax rate	15.0%
Capital expenditures	31.25
Working capital	0
WACC	8.5%
Growth	4.0%

(b) Use the numbers above to compute EV/EBITDA

(c) Value (EV) can also be expressed

$$\text{EV} = \frac{\text{EBITDA}(1-t) + \text{depreciation} \cdot t - \text{capital expenditures} - \Delta \cdot \text{working capital}}{\text{WACC} - \text{growth}}$$

Compute EV using this equation, and then check that this number is consistent with your estimate in the previous question by multiplying the EV/EBITDA ratio with EBITDA

(d) Then compute the value-to-sales multiple. Since we know the warranted EV/EBITDA multiple, you may obtain EV/Sales by multiplying with EBITDA/Sales,

(e) Lastly compute the P/E multiple. In this case where dere is no debt, NOPAT and earnings are the same, and earnings equals EBIT$\cdot(1-t)$. As usual, you may presume value and price are synonymous.

Solution:

(a)

$$\begin{aligned}
\text{EV} &= \frac{\text{Free cash flow}}{\text{Cost of capital} - \text{growth}} \\
&= \frac{\text{EBIT}(1-t) - (\text{capital expenditures} - \text{depreciation}) - \Delta \text{ in working capital}}{\text{Cost of capital} - g} \\
&= \frac{\text{EBITDA }(1-t) + \text{depreciation} \cdot t - \text{capital expenditures} - \Delta \text{ in working capital}}{\text{WACC} - g}
\end{aligned}$$

so

$$\frac{\text{EV}}{\text{EBITDA}} = \frac{1-t}{\text{WACC} \ - g} + \frac{\text{Depreciation} \cdot t/\text{EBITDA}}{\text{WACC} \ - g} - \frac{\text{capital expenditures/EBITDA}}{\text{WACC} \ - g} - \frac{\Delta \text{ in working capital/EBITDA}}{\text{WACC} \ - g}$$

(b)

$$\frac{\text{EV}}{\text{EBITDA}} = \frac{0.85}{0.085 - 0.040} + \frac{\frac{25 \cdot 0.15}{125}}{0.085 - 0.040} - \frac{\frac{31.25}{125}}{0.085 - 0.040} - 0$$
$$= \frac{0.85}{0.45} + \frac{0.03}{0.45} - \frac{0.25}{0.45} = 14.0$$

(c)

$$\text{EV} = \frac{100(1 - .15) - (31.25 - \$25) - 0}{.085 - .04} = \frac{78.75}{.045} = 1750$$

Just for later, instructional purposes: to check

$$\text{EV} = \frac{\text{EV}}{\text{EBITDA}} \cdot \text{EBITDA} \ = 14.0 \cdot 125 = 1750$$

(d) Since we know the EV/EBITDA multiple, it is a short step to calculate the EV/Sales multiple

$$\frac{\text{EV}}{\text{Sales}} = \frac{\text{EV}}{\text{EBITDA}} \cdot \frac{\text{EBITDA}}{\text{Sales}} = 14.0 \cdot 0.25 = 3.5$$

(e) Since there is no debt, earnings are EBIT net of taxes, and the price-earnings-ratio is

$$\text{P/E} = \frac{1750}{100 \cdot (1 - 0.15)} = 20.6$$

In reality, the Gordon growth model is too simplistic to capture the value of most businesses. But these questions show how valuation multiples tie to the core drivers of value and to one another.

(20 points) 5. *Investments in risky technologies*

(a) Consider the following investment case ("Case A") with a highly risky technology:

- Cost setting up a production facility: 100M
- Construction time: 1 year
- After the facility has been constructed, the owners will know whether the technology is a success or not:
  - With 50% probability the production facility will generate a net cash flow of 25M in perpetuity
  - With 50% probability the production facility will generate a net cash flow of 1.5M in perpetuity
- The ordinary tax rate is 20% on net cash flows.
- Given the riskiness of the project, the required rate of return is 10%

What is the expected after-tax NPV of the project? Should the investors build the production facility or should they shelve their plans for good?

(b) Hypothetical outcome: the entrepreneurs decided to build the production facility, and the technology turned out to be a success. After one year of operations, the government observes that the business is highly profitable, and that, since all investments have already been made, it is not physically mobile. They suddenly announced that they would levy an additional 40 percentage points "ground rent tax" on all future net cash flows on this facility.

i. Will the business continue its operation?

ii. If it had been known *ex ante* that the government would levy an additional 40 percentage points "ground rent tax" if the project was a success, what would the expected NPV have been? Would the facility have been built? You may assume that the required rate of return on capital was unchanged.

(c) Consider another investment case, a few years later, ("Case B") with another highly risky technology. They have observed the political fallout after "Case A" and the additional tax that was levied on that industry.

- Cost setting up a production facility: 150M
- Construction time: 1 year
- After the facility has been constructed, the owners will know whether the technology is a success or not:
  - With 50% probability the production facility will generate a net cash flow of 40M in perpetuity
  - With 50% probability the production facility will generate a net cash flow of 1M in perpetuity
- The ordinary tax rate is 20% on net cash flows.

- The entrepreneurs assess the political risk and assume that with 50% probability they will be levied an additional 40 percentage points additional "ground rent tax" if the technology turns out to be a success
- In absence of political risk, the required rate of return would still have been 10%

First ignoring political risk, what is the expected NPV?

(d) Compute the additional risk premia investors would require when there is political risk. (Hint: the additional risk premium may be computed analogously to how yields are computed on risky bonds, ie. bonds with a strictly positive default probability).

Solution:

(a)

$$\begin{aligned} \text{NPV} &= -100 + \frac{0.5 \cdot 25 \cdot (1 - .2) + 0.5 \cdot 1.5 \cdot (1 - .2)}{0.10} \\ &= -100 + \frac{10 + 0.6}{0.10} = 6 > 0 \end{aligned}$$

(b) i. Yes, they will still make an *ex post* profit

ii. No, the expected NPV would then have been negative:

$$\begin{aligned} \text{NPV} &= -100 + \frac{0.5 \cdot 25 \cdot (1 - .6) + 0.5 \cdot 1.5 \cdot (1 - .2)}{0.10} \\ &= -100 + \frac{5 + 0.6}{0.10} = -44 < 0 \end{aligned}$$

(c) Ignoring political risk

$$\begin{aligned} \text{NPV} &= -150 + \frac{0.5 \cdot 40 \cdot (1 - .2) + 0.5 \cdot 1.0 \cdot (1 - .2)}{0.10} \\ &= -150 + \frac{16 + 0.4}{0.10} = 14 > 0 \end{aligned}$$

(d) Solving for x

$$\begin{aligned} -150 + \frac{16 + 0.4}{x} &= -150 + \frac{0.5 \cdot [0.5 \cdot 40 \cdot (1 - .6) + 0.5 \cdot 40 \cdot (1 - .2)] + 0.4}{0.10} \\ &= -150 + \frac{0.5 \cdot [8 + 16] + 0.4}{0.10} \\ &= -150 + \frac{12 + 0.4}{0.10} \end{aligned}$$

so

$$x = \frac{16.4}{124} = 0.1322 = 13.22\%$$

ie. the additional political risk premium would be 3.22 percentage points.

(30 points) 6. *Asset pricing, options, and the CAPM*

Use a one-period binomial model with two future states at time 1, up and down. The market index, which at time 0 has value $S_0 = 500$, takes the values $S_u = 624$ and $S_d = 468$ in the states "up" and "down", respectively. Assume that the gross risk-free interest rate is $R^f = 1.04$.

(a) Check whether the state prices, q_u and q_d for states up and down, respectively, in this model are

$$q_u = \frac{25}{78}, \quad q_u = \frac{25}{39}$$

(b) A put option on the market index with exercise price K has time-1 payoff $\max\{K - S_1, 0\}$, where $S_1 = S_u$ in state up, and $S_1 = S_d$ in state down. Assume that $K = 507$.

Calculate P_0, the time-0 value of the put option, from the state prices.

(c) Let m_u and m_d be the value of the stochastic discount factor in the states up and down, respectively.

Explain why

$$q_u = \pi m_u, \qquad \text{and} \qquad q_d = (1 - \pi) m_d$$

where π is the probability for state up.

Assume that $\pi = \frac{5}{13}$.

(d) Calculate the expected gross return $\mathbf{E}[R_m]$ of the market index.

(e) Calculate m_u and m_d.

(f) Calculate the time-0 price of the put option from the formula

$$P_0 = \mathbf{E}[mX]$$

where $m = m_u$ in state up, and $m = m_d$ in state down. Also, $X = \max(K - S_1, 0)$.

(g) If the stochastic discount function can be written as a linear function of the market return, the CAPM is equivalent to the state price/stochastic discount factor approach. I.e, if we can find constants a and b such that

$$a + bR_m^u = m_u$$

and

$$a + bR_m^d = m_d$$

then the CAPM produces the same prices as the other two approaches. Please compute a and b.

Denote the time-0 price of the put option calculated by CAPM by q.

In the following four questions, the answers should be stated in terms of the state prices q.

(h) Calculate the gross return R_i of the put option as a function of the unknown time-0 price q.

(i) Calculate $\text{Cov}(R_m, R_i)$, the covariance between R_m and R_i, and $\text{Var}(R_m)$, the variance of the return of the market index.
Reminder 1: $\text{Var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$.
Reminder 2: $\text{Cov}(X, Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y]$.

(j) Calculate the β of the put option.
Reminder:

$$\beta = \frac{\text{Cov}(R_m, R_i)}{\text{Var}(R_m)}$$

(k) Calculate $\mathbf{E}[R_i]$, the expected return of the option from CAPM.
Reminder: CAPM states that

$$\mathbf{E}[R_i] = R^f + \beta\left[\mathbf{E}[R_m] - R^f\right]$$

By definition, $\mathbf{E}[X] = q\mathbf{E}[R_i]$, where $X = \max(K - S_1, 0)$ is the (random) payoff of the put option.

(l) Calculate q by equating $\mathbf{E}[R_i] = \frac{\mathbf{E}[X]}{q}$ from the above equation with the expression for $\mathbf{E}[R_i]$ from CAPM in (k). Comment whether $q = P_0$.

Solution:

(a) We know that buying both the two states is the same as buying a risk-free bond that pays off in both states so

$$q_u + q_d = \frac{1}{R^f} = \frac{1}{1.04}$$

We also know that the value of the two possible future states have to equal the value, ie. the market index, today

$$624\pi_u + 468\pi_d = 500$$

We have two equations in two unknowns and can check that

$$q_u = \frac{25}{78}, \quad q_d = \frac{25}{39}$$

solve these two equations

(b) The state prices are the prices of one unit in consumption in the two states $\{u, d\}$, respectively, so

$$P_0 = q_u P_u + q_d P_d = 25$$

We also have

$$P_u = \max(507 - 624, 0) = 0$$
$$P_d = \max(507 - 468, 0) = 39$$

hence

$$P_0 = q_u P_u + q_d P_d = \frac{25}{78} \cdot 0 + \frac{25}{39} \cdot 39 = 25$$

(c) Claim q_u pays 1 in state "up" and 0 in state "down". From $p = \mathbf{E}[mX]$,

$$q_u = \pi \cdot m_u \cdot 1 + (1 - \pi) \cdot m_d \cdot 0 = \pi \cdot m_u$$

Claim q_d pays 1 in state "down" and 0 in state "up". From $p = \mathbf{E}[mX]$,

$$q_d = \pi \cdot m_u \cdot 0 + (1 - \pi) \cdot m_d \cdot 1 = (1 - \pi)m_d$$

(d)

$$\mathbf{E}[R_m] = \pi \cdot R_m^u + (1 - \pi) \cdot R_m^d = \frac{5}{13} \cdot \frac{624}{500} + \frac{8}{13} \cdot \frac{460}{500} = 1.056$$

(e) From (c)

$$q_u = \pi \cdot m_u \quad \text{and } q_d = (1 - \pi) \cdot m_d$$

so

$$m_u = \frac{q_u}{\pi} = \frac{5}{6}$$
$$m_d = \frac{q_d}{1 - \pi} = \frac{25}{24}$$

(f)

$$P_0 = \mathbf{E}[mX] = \pi \cdot m_u \cdot P_u + (1 - \pi) \cdot m_d \cdot P_d$$

$P_u = 0$, and $P_d = 39$ from (b). Thus,

$$P_0 = \mathbf{E}[mX] = \frac{5}{13} \cdot 56 \cdot 0 + \frac{8}{13} \cdot \frac{25}{24} \cdot 39 = 25$$

(g) If the stochastic discount function can be written as a linear function of the market return, the CAPM is equivalent to the state price/stochastic discount factor approach. I.e, if we can find constants a and b such that

$$a + bR_m^u = m_u$$

and

$$a + bR_m^d = m_d$$

then the CAPM produces the same prices as the other two approaches.

The above two equations have two unknowns, so a solution (a, b) exists.

It is easy to show that the solution is $a = 5/3$ and $b = -625/936 = -0.667735$.

Thus, we will get the same price for the option if we price it by CAPM as we have got in (b) and (f), i.e. $P_0 = 25$.

(h) $R_i^u = 0$ and $R_i^d = \frac{39}{q}$. We can, e.g., write $R_i = \frac{39}{q}$ if $S_1 = S_d$

(i)

$$\mathbf{E}\left[R_m^2\right] = p \cdot \left(R_m^u\right)^2 + (1-p)\left(R_m^d\right)^2 = 1.138176$$
$$\text{Var}\left(R_m\right) = 1.138176 - 1.056^2 = 0.02304$$
$$\mathbf{E}\left(R_m R_i\right) = p \cdot R_m^u R_i^u + (1-p) R_m^d R_i^d = \frac{22.464}{q}$$
$$\text{Cov}\left(R_m, R_i\right) = \frac{22.464}{q} - 1.056\frac{39}{q}8/13 = \frac{-2.88}{q}$$

(j)

$$\beta = \frac{\text{Cov}\left(R_m, R_i\right)}{\text{Var}\left(R_m\right)} = -\frac{125}{q}$$

(k)

$$E\left[R_i\right] = R^f + \beta\left[\mathbf{E}\left[R_m\right] - R^f\right] = 1.04 - \frac{2}{q}$$

(l)

$$\mathbf{E}\left[R_i\right] = \frac{\mathbf{E}[X]}{q} = 1.04 - \frac{2}{q}$$

or, by inserting for $\mathbf{E}[X]$ and multiplying all terms by q,

$$\frac{8}{13} \cdot 39 = 1.04q - 2$$

or, as we knew from (g)

$$q = \frac{26}{1.04} = 25 = P_0$$