Derivation of CAPM formula, contd.

Use the formula:

$$\frac{d\mu}{d\sigma}\frac{\partial\sigma}{\partial a} = \frac{\partial\mu}{\partial a} \iff \frac{d\mu}{d\sigma} = \frac{\frac{\partial\mu}{\partial a}}{\frac{\partial\sigma}{\partial a}}.$$

Use partial derivatives just found, evaluate at $a = 0$:

$$\left.\frac{\partial\sigma}{\partial a}\right|_{a=0} = \frac{\sigma_{jM} - \sigma_M^2}{\sigma_M}.$$

Plug in and find:

$$\left.\frac{d\mu}{d\sigma}\right|_{a=0} = \frac{\mu_j - \mu_M}{(\sigma_{jM} - \sigma_M^2)/\sigma_M}.$$

This slope of small hyperbola must equal slope of CML:

$$\frac{\mu_j - \mu_M}{(\sigma_{jM} - \sigma_M^2)/\sigma_M} = \frac{\mu_M - r_f}{\sigma_M} \iff \mu_j = r_f + (\mu_M - r_f)\frac{\sigma_{jM}}{\sigma_M^2}.$$

Known as the CAPM equation or the Security Market Line.

Define $\beta_j \equiv \frac{\sigma_{jM}}{\sigma_M^2}$. Then rewrite as

$$E(\tilde{r}_j) - r_f = \beta_j(E(\tilde{r}_M) - r_f),$$

"the expected *excess* rate of return on asset j equals its beta times the expected excess rate of return on the market portfolio."

Illustrating the Security Market Line

$$\mu_j = r_f + \beta_j(\mu_M - r_f).$$

- All securities are located on the line.
- Also any portfolio of m securities. Show for $m = 2$:

$$\mu_p = a\mu_i + (1-a)\mu_j = a[r_f + \beta_i(\mu_M - r_f)] + (1-a)[r_f + \beta_j(\mu_M - r_f)]$$

$$= r_f + [a\beta_i + (1-a)\beta_j](\mu_M - r_f)$$

$$= r_f + \left[a\frac{\text{cov}(\tilde{r}_i, \tilde{r}_M)}{\sigma_M^2} + (1-a)\frac{\text{cov}(\tilde{r}_j, \tilde{r}_M)}{\sigma_M^2}\right](\mu_M - r_f)$$

$$= r_f + \frac{\text{cov}(a\tilde{r}_i + (1-a)\tilde{r}_j, \tilde{r}_M)}{\sigma_M^2}(\mu_M - r_f)$$

$$= r_f + \frac{\text{cov}(\tilde{r}_p, \tilde{r}_M)}{\sigma_M^2}(\mu_M - r_f) = r_f + \beta_p(\mu_M - r_f).$$

- β_p is a value-weighted average of β_i and β_j.

Interpretation of CAPM equation

$$E(\tilde{r}_j) = r_f + \frac{\sigma_{jM}}{\sigma_M^2}(E(\tilde{r}_M) - r_f).$$

Verbal interpretation:

> The expected rate of return on any asset depends on only one characteristic of that asset, namely its rate of return's covariance with the rate of return on the market portfolio.
>
> The expected rate of return is equal to the risk free interest rate plus a term which depends on a measure of risk. (Higher risk means higher expected rate of return.) The relevant measure of risk is the asset's beta. This is multiplied with the expected excess rate of return on the market portfolio.

- Obvious: Does not say what r_j will be. Only $E(\tilde{r}_j)$.
- Risk measure depends on covariance because the covariance determines how much that asset will contribute to the risk of the agent's portfolio.
- This is true for any agent, since all hold the same risky portfolio.

Interpretation, contd.

- Observe $\beta_M = 1$.
- Observe $\beta_j = \rho_{jM}\sigma_j/\sigma_M$.
- May have $\sigma_j > \sigma_M$, and ρ_{jM} close to 1.
- Thus possible to have $\beta_j > 1$ for some assets.
- May also have $\text{cov}(\tilde{r}_j, \tilde{r}_M) < 0,\ \beta_j < 0$.
- Not very common in practice.
- Such assets contribute to reducing σ_M (when included in M).
- Valuable to investors. (Question: What does this imply for value at $t = 0$? For expected rate of return?)

Avoid confusion

- Do not confuse the following two:
- The *Capital Market Line:*
  - Efficient set given 1 risk free and n risky assets.
  - Relevant for choice between alternative portfolios.
  - Drawn in (σ_p, μ_p) diagram.
  - A ray starting at $(0, r_f)$ in that diagram.
- The *Security Market Line:*
  - Location of *all* traded assets in equilibrium.
  - Also location of any portfolio of these assets.
  - Not relevant for choice between assets which are already traded, so that equilibrium prices are observable.
  - But relevant if equilibrium price at $t = 0$ is unknown.
  - Drawn in (β_j, μ_j) diagram.
  - A line through $(0, r_f)$ in that diagram.

Other forms of the CAPM equation

Remember: $1 + \tilde{r}_j = \tilde{p}_{j1}/p_{j0}$. Rewrite:

$$\frac{E(\tilde{p}_{j1})}{p_{j0}} - 1 = r_f + \frac{\mu_M - r_f}{\sigma_M^2} \operatorname{cov}\left(\frac{\tilde{p}_{j1}}{p_{j0}} - 1, \tilde{r}_M\right)$$

$$= r_f + \frac{\mu_M - r_f}{\sigma_M^2} \left[\operatorname{cov}\left(\frac{\tilde{p}_{j1}}{p_{j0}}, \tilde{r}_M\right) - \operatorname{cov}(1, \tilde{r}_M)\right]$$

$$\iff \quad \frac{E(\tilde{p}_{j1})}{p_{j0}} = 1 + r_f + \lambda \operatorname{cov}\left(\frac{\tilde{p}_{j1}}{p_{j0}}, \tilde{r}_M\right)$$

$$= 1 + r_f + \lambda \operatorname{cov}\left(\frac{1}{p_{j0}}\tilde{p}_{j1}, \tilde{r}_M\right),$$

with λ defined as $(\mu_M - r_f)/\sigma_M^2$.

$\frac{1}{p_{j0}}$ is deterministic, and can be factored out of the covariance.

$$\iff \quad \frac{E(\tilde{p}_{j1})}{p_{j0}} = 1 + r_f + \frac{\lambda \operatorname{cov}(\tilde{p}_{j1}, \tilde{r}_M)}{p_{j0}}.$$

Multiply both sides by

$$\frac{p_{j0}}{1 + r_f}$$

and rearrange terms to get

$$p_{j0} = \frac{1}{1 + r_f}\left[E(\tilde{p_{j1}}) - \lambda \operatorname{cov}(\tilde{p}_{j1}, \tilde{r}_M)\right].$$

Expression in square bracket is called the *certainty equivalent* (in the CAPM sense) of $\tilde{p}_{j1}$. Price today is present value of expression, just as if that would be received with certainty one period into the future. *Not* what is called certainty equivalent in expected utility theory.

Other forms of the CAPM expression, contd.

Can also rewrite as

$$p_{j0} = \frac{E(\tilde{p}_{j1})}{1 + r_f + \lambda \operatorname{cov}(\tilde{r}_j, \tilde{r}_M)}.$$

(Prove yourself how to arrive at this!)

- Like previous form: p_{j0} on left-hand side.
- But here: Right-hand side is "expected present value."
- However: *Risk-adjusted discount rate,* RADR.
- Obtained by adding something to r_f.
- Risk-adjustment again depends on λ and covariance.
- Observe difference in covariance expressions!
- Previously: $\operatorname{cov}(\tilde{p}_{j1}, \tilde{r}_M)$.
- Now: $\operatorname{cov}(\tilde{r}_j, \tilde{r}_M) \equiv \operatorname{cov}(\tilde{p}_{j1}/p_{j0}, \tilde{r}_M)$.
- If need to solve for p_{j0}, the formula on top of this page is less useful: It has p_{j0} also on right-hand side, "hidden" in covariance term.
- Use instead expression on previous page.

CAPM: Implications, applications

- For a portfolio: Variance the relevant risk measure.
- For each security: Covariance with $\tilde{r}_M$ the relevant r. m.
- Reason: Covariance measures contribution to portfolio variance.
- Generally: Covariance with each agent's marginal utility.
- But when mean-var preferences: Covariance with agent's wealth.
- In equilibrium: All have same risky portfolio (composition).
- Thus covariance with $\tilde{r}_M$ relevant for everyone.
- Implication: Adding noise does not matter:
- Suppose $\tilde{p}_{h1} = \tilde{p}_{j1} + \tilde{\varepsilon}$, where $E(\tilde{\varepsilon}) = 0$ and $\tilde{\varepsilon}$ is stochastically independent of $(\tilde{p}_{j1}, \tilde{r}_M)$. Can show $p_{h0} = p_{j0}$:

$$p_{h0} = \frac{1}{1+r_f}\left[E(\tilde{p}_{j1} + \tilde{\varepsilon}) - \lambda \operatorname{cov}(\tilde{p}_{j1} + \tilde{\varepsilon}, \tilde{r}_M)\right]$$

$$= \frac{1}{1+r_f}\left[E(\tilde{p}_{j1}) + E(\tilde{\varepsilon}) - \lambda[\operatorname{cov}(\tilde{p}_{j1}, \tilde{r}_M) + \operatorname{cov}(\tilde{\varepsilon}, \tilde{r}_M)]\right] = p_{j0}.$$

- At this point: Do *not* get confused by ρ_{jM}.
- Common confusion: Since $\sigma_{jM} = \rho_{jM}\sigma_j\sigma_M$, some believe that a higher σ_j leads to a higher σ_{jM}.
- Have just shown a case where this is not true; instead:

$$\rho_{hM} = \frac{\sigma_{hM}}{\sigma_h\sigma_M} = \frac{\sigma_{jM}}{\sigma_h\sigma_M} < \frac{\sigma_{jM}}{\sigma_j\sigma_M} = \rho_{jM}.$$

CAPM terminology: Systematic vs. unsystematic risk

Define $\tilde{\varepsilon}_j$ through the equation

$$\tilde{r}_j = r_f + \beta_j \left[\tilde{r}_M - r_f\right] + \tilde{\varepsilon}_j.$$

(This is *not* the ε variable used on the previous page.)

Then the CAPM equation

$$E(\tilde{r}_j) = r_f + \beta_j \left[E(\tilde{r}_M) - r_f\right]$$

implies that $E(\tilde{\varepsilon}_j) = 0$, and that

$$\text{cov}(\tilde{\varepsilon}_j, \tilde{r}_M) = \text{cov}(\tilde{r}_j, \tilde{r}_M) - \beta_j \,\text{var}(\tilde{r}_M) = \sigma_{jM} - \beta_j \sigma_M^2 = 0.$$

This allows us to split σ_j^2 in two parts:

$$\text{var}(\tilde{r}_j) \equiv \sigma_j^2 = \beta_j^2 \sigma_M^2 + \sigma_{\varepsilon_j}^2 = \beta_j \sigma_{jM} + \sigma_{\varepsilon_j}^2.$$

First term called *systematic risk.* This is reflected in the market valuation. (More general term: *Relevant risk* or *covariance risk.*)

Second term called *unsystematic risk.* As we have seen, it is not reflected in market valuation. (More general term: *Irrelevant risk.*)

The sum of the two called *total risk* or *variance risk.* This *is* relevant for portfolios, evaluated for being the total wealth of someone, but not for individual securities, to be combined with other securities in portfolios.

Valuation of firms, value additivity

$$p_{i0} = \frac{1}{1+r_f}\left[E(\tilde{p}_{i1}) - \lambda \operatorname{cov}(\tilde{p}_{i1}, \tilde{r}_M)\right] \equiv V(\tilde{p}_{i1}),$$

defines valuation function $V()$, given some $r_f, \tilde{r}_M$.

- $\tilde{p}_{i1}$ is value of share at $t = 1$, including dividends.
- Consider firm financed 100% by equity (no debt).
- Firm's net cash flow at $t = 1$ goes to shareholders.
- Plug in that net cash flow instead of $\tilde{p}_{i1}$.
- Then formula gives value of *all shares* in firm.
- What if $\tilde{p}_{i1} = a\tilde{p}_{j1} + b\tilde{p}_{k1}$? ($a > 0,\ b > 0$ constants.)
- Linearity of $E()$ and cov() implies $p_{i0} = ap_{j0} + bp_{k0}$:

$$p_{i0} = \frac{1}{1+r_f}\left[E(\tilde{p}_{i1}) - \lambda \operatorname{cov}(\tilde{p}_{i1}, \tilde{r}_M)\right]$$

$$= \frac{1}{1+r_f}\left[E(a\tilde{p}_{j1} + b\tilde{p}_{k1}) - \lambda \operatorname{cov}(a\tilde{p}_{j1} + b\tilde{p}_{k1}, \tilde{r}_M)\right]$$

$$= \frac{1}{1+r_f}\left[aE(\tilde{p}_{j1}) + bE(\tilde{p}_{k1}) - \lambda[a \operatorname{cov}(\tilde{p}_{j1}, \tilde{r}_M) + b \operatorname{cov}(\tilde{p}_{k1}, \tilde{r}_M)]\right]$$

$$= ap_{j0} + bp_{k0}.$$

- Diversification is no justification for mergers.
- Diversification can be done by shareholders.

Investment project's rate of return

- Consider (potential) real investment project:
  - Outlay I at $t = 0$.
  - Revenue $\tilde{p}_{I1}$ at $t = 1$.
- Project value? Should project be undertaken?
- Assume 100% equity financed. (Assume separate firm?)
- Project's rate of return is $(\tilde{p}_{I1} - I)/I$.
- If use of restricted technology or resources: No reason for SML equation to hold for this rate of return.
- May earn above-normal expected rate of return.
- However: Valuation of $\tilde{p}_{I1}$ possible:

$$p_{I0} = V(\tilde{p}_{I1}) = \frac{1}{1 + r_f} [E(\tilde{p}_{I1}) - \lambda \operatorname{cov}(\tilde{p}_{I1}, \tilde{r}_M)]$$

  defines p_{I0} *independently of* I.
- If $p_{I0} > I$, undertake project. Net value $p_{I0} - I$.
- If $p_{I0} < I$, drop project. Net value of opportunity is 0. Net value of *having to* undertake project is $p_{I0} - I$, negative.
- Competition and free entry $\Rightarrow$ $p_{I0} = I$ in long run, through increased supply, lower $E(\tilde{p}_{I1})$.

Project valuation, contd.

- If claim to $\tilde{p}_{I1}$ costs p_{I0}, this is equilibrium price.
- Project is then on security market line (SML).
- If project available at different cost I: Not at SML.

Project's β_k, must satisfy the equation

$$E(\frac{\tilde{p}_{I1}}{p_{I0}}) = 1 + r_f + [\mu_M - r_f]\beta_k$$

$$= 1 + r_f + [\mu_M - r_f]\,\mathrm{cov}(\frac{\tilde{p}_{I1}}{p_{I0}}, \tilde{r}_M)/\sigma_M^2.$$

Expressed in terms of exogenous variables (eliminating p_{I0}) this becomes

$$\beta_k = \frac{(1 + r_f)}{\sigma_M^2 \frac{E(\tilde{p}_{I1})}{\mathrm{cov}(\tilde{p}_{I1}, \tilde{r}_M)} - \mu_M + r_f},$$

independent of I. Only if $I = p_{I0} = V(\tilde{p}_{I1})$, will the project rate of return $\tilde{p}_{I1}/I - 1$ satisfy the CAPM equation with β_k.