## Replicating portfolios

- Buy a number of shares, $\Delta$, and invest $B$ in bonds.
- Outlay for portfolio today is $S\Delta + B$.
- Tree shows possible values one period later.

$$\begin{tikzcd}
 & uS\Delta + e^r B \\
S\Delta + B \arrow[ur, no head, "p^*"] \arrow[dr, no head, "1-p^*"'] & \\
 & dS\Delta + e^r B
\end{tikzcd}$$

- Choose $\Delta, B$ so that portfolio *replicates* call.
- "Replicate" (*duplisere*) means mimick, behave like.
- Two equations:

$$\begin{aligned} uS\Delta + e^r B &= c_u, \\ dS\Delta + e^r B &= c_d, \end{aligned}$$

with solutions

$$\Delta = \frac{c_u - c_d}{(u-d)S}, \quad B = \frac{uc_d - dc_u}{(u-d)e^r}.$$

## Replicating portfolio, contd.

- $(\Delta, B)$ gives same values as option in both states.
- Also called option's *equivalent* portfolio.
- Must have same value now, $c = S\Delta + B$

  $$= \frac{(c_u - c_d)e^r + uc_d - dc_u}{(u-d)e^r} = \frac{(e^r - d)c_u + (u - e^r)c_d}{(u-d)e^r}.$$

  Define $p \equiv (e^r - d)/(u - d)$. (Observe $d \leq e^r \leq u \Rightarrow 0 \leq p \leq 1$.) Rewrite formula as

  $$c = \frac{pc_u + (1-p)c_d}{e^r}.$$

- Show $c = S\Delta + B$ by absence-of-arbitrage.
- If observe $c_{\text{obs}} < S\Delta + B$: Buy option, sell pf.
- Cash in $-c_{\text{obs}} + S\Delta + B > 0$ now.
- Keep until expiration.
- In both states, net value is then zero.
- If observe $c_{\text{obs}} > S\Delta + B$: Buy pf., write option.
- Cash in $c_{\text{obs}} - S\Delta - B > 0$ now.
- Keep until expiration.
- In both states, net value is then zero.

## Comments on one-period formula

$$c = \frac{pc_u + (1-p)c_d}{e^r} \quad \text{with} \quad p \equiv \frac{e^r - d}{u - d}$$

- Based on binomial model for share prices.
- Formula independent of $p^*$. If all believe in same $u, d$, may believe in different $p^*$'s, and still agree on call option value.
- $p^*$ important for $E(S_T)$.
- (Different opinions about) $E(S_T)$ do not affect option value.

## Absence-of-arbitrage proof for American option

- Need extra argument if option is American.
- If you write and sell option, buyer may exercise now.
- Happens if $C_{\text{obs}} < S - K$.
- Then you (the writer, issuer) lose $S - K - C_{\text{obs}}$.
- Seems that for American option:

$$C = \begin{cases} S\Delta + B & \text{if } S\Delta + B > S - K, \\ S - K & \text{if } S\Delta + B \le S - K. \end{cases}$$

- But show: $[e^r > 1 \text{ and } D = 0] \;\Rightarrow\; S\Delta + B > S - K$

## Proof of no early exercise of American option

Distinguish three cases:

1. $uS \leq K$. Then $C_u = C_d = 0$ and $S < K$, so
$$\frac{pC_u + (1-p)C_d}{e^r} = 0 > S - K.$$
2. $dS \leq K < uS$. Then $C_d = 0, C_u = uS - K$, and
$$\frac{pC_u + (1-p)C_d}{e^r} = \frac{p(uS-K)}{e^r}.$$
Need to show that $e^r > 1$ and $K > dS$ implies
$$\frac{p(uS-K)}{e^r} > S - K,$$
which can be rewritten
$$(e^r - p)K > (e^r - pu)S = \left(e^r - \frac{e^r - d}{u-d}u\right)S = (1-p)dS,$$
which is true, since $e^r - p > 1 - p$ and $K > dS$.
3. $K \leq dS$. Then $C_u = uS - K,\ C_d = dS - K$, option will for sure be exercised, giving
$$C = \frac{1}{e^r}[p(uS-K) + (1-p)(dS-K)] = S - \frac{K}{e^r} > S - K.$$

In all three cases, $S\Delta + B > S - K$, implying no early exercise.

**Two facts (exercises for you):**

Left for you to show:

1. Must have $d \leq e^r \leq u$. (Use absence-of-arbitrage proof.)
2. From formula, have $B \leq 0$. (Consider the three cases.)

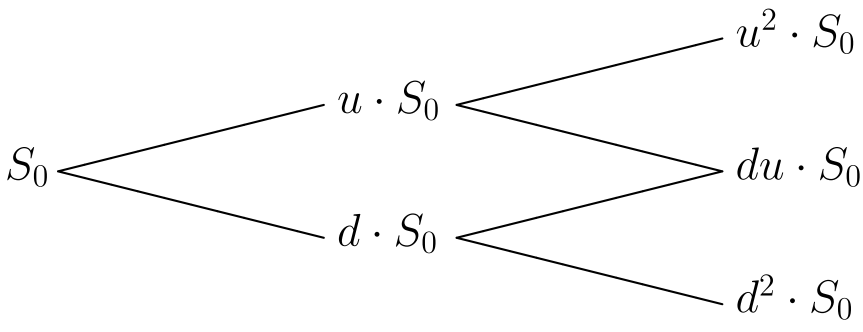
Interpretation of $d \leq e^r \leq u$: This implies

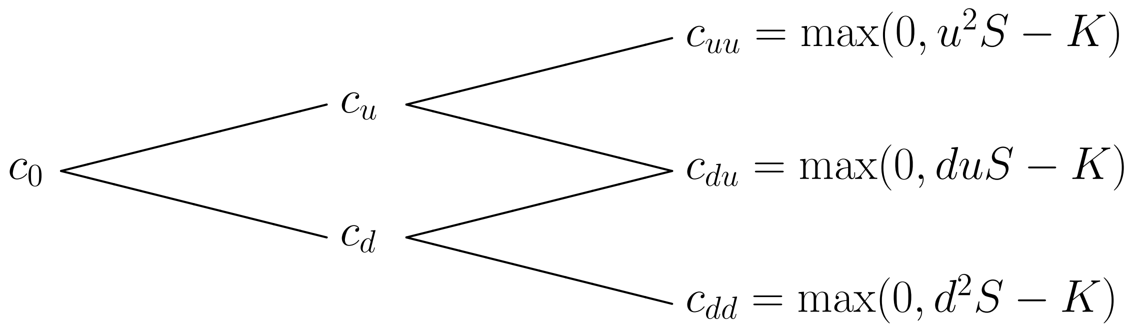
$$p \equiv \frac{e^r - d}{u - d} \in [0, 1].$$

Could this be interpreted as a probability, perhaps? Will return to this question soon.

Interpretation of $B \leq 0$: Recall that $c \geq 0$. Thus replicating portfolio $S\Delta + B$ has positive value. Since $B \leq 0$, the portfolio includes a short sale of bonds, i.e., borrowing at the risk free interest rate. Total outlay is $c$, but *more than* $c$ is invested in shares. Evaluated in isolation, the pf. is thus more risky than investing in the shares only: It consists in borrowing $|B|$ and investing $c+|B|$ in shares. (The fact that call options have higher risk (both systematic and total) than the underlying shares, is true generally, but in this course we will only look at it here, for the one-period case in the binomial model.)

## Extension to two periods




- Consider option with expiration two periods from now.
- Want today's (t=0) call option value.
- *Redefine:* $c_u, c_d$ and $c_0$ get new meanings.
- Solution extends idea of replicating portfolio.
- Solve problem backward in time.
- First: Find replicating portfolios at $t = 1$:
  - For the upper node with $S_1 = u \cdot S_0$.
  - For the lower node with $S_1 = d \cdot S_0$.

## Extension to two periods, contd.

- Absence-of-arbitrage argument shows:
- Value at $S_1 = u \cdot S_0$ node is
$$c_u = \frac{pc_{uu} + (1-p)c_{du}}{e^r}.$$
- Value at $S_1 = d \cdot S_0$ node is
$$c_d = \frac{pc_{du} + (1-p)c_{dd}}{e^r}.$$
- For any $K$ these numbers are known.
- *Not* the same $c_u, c_d, c_0$ as in one-period problem.
- Find $t = 0$ value: Construct new replicating pf.
- Portfolio at $t = 0$ which ends up at $t = 1$ as $c_u, c_d$ resp.
- Value of that pf. is
$$c_0 = \frac{pc_u + (1-p)c_d}{e^r} = \frac{p^2 c_{uu} + 2p(1-p)c_{du} + (1-p)^2 c_{du}}{e^{2r}}.$$
- Three different replicating portfolios in this problem.
- Call them $(\Delta_0, B_0),\ (\Delta_u, B_u),\ (\Delta_d, B_d)$.
- Cannot make one replic. pf. at $c_0$ and keep until $t = 2$.
- Extension called *replicating portfolio strategy.*
- "Strategy" means plan describing actions to be taken contingent on arriving information, here $S_1$.

## Extension to two periods, contd.

- For completeness, some formulae:

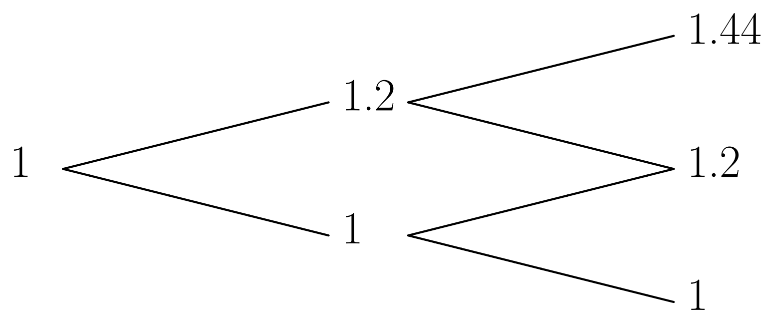
$$\Delta_u = \frac{c_{uu} - c_{du}}{(u-d)uS_0}, \quad B_u = \frac{uc_{du} - dc_{uu}}{(u-d)e^r},$$

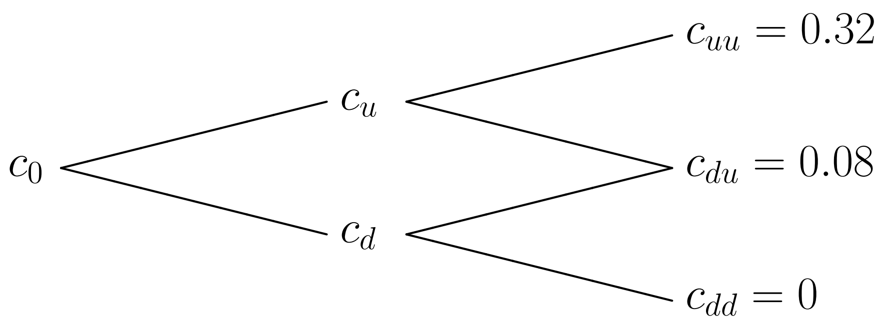
$$\Delta_d = \frac{c_{du} - c_{dd}}{(u-d)dS_0}, \quad B_d = \frac{uc_{dd} - dc_{du}}{(u-d)e^r}.$$

- The replicating portfolio strategy is *self financing*:
- The replicating portfolio must be changed at $t = 1$.
- Whether $S_1 = uS_0$ or $S_1 = dS_0$, this change costs exactly zero.

## Two-period call option example

$e^r = 1.1,\ u = 1.2,\ d = 1;\ p = \frac{e^r - d}{u-d} = \frac{1}{2},\ S_0 = 1,\ K = 1.12.$



## Two-period call option example

- In example: Call option with expiration at $t = 2$.
- Want to find option's value at $t = 0$. Formula.
- Using formulae, may fill in $c$ values in tree:

$$c_u = \frac{pc_{uu} + (1-p)c_{du}}{e^r} = \frac{0.2}{e^r} = \frac{2}{11} \approx 0.18,$$

$$c_d = \frac{pc_{du}}{e^r} = \frac{0.04}{e^r} = \frac{2}{55} \approx 0.036,$$

$$c_0 = \frac{pc_u + (1-p)c_d}{e^r} = \frac{\frac{1}{2}\cdot\frac{2}{11} + \frac{1}{2}\cdot\frac{2}{55}}{e^r} = \frac{12}{121} \approx 0.10.$$

$$\begin{tikzcd}[arrows=dash]
 & & c_{uu} = 0.32 \\
 & c_u \approx 0.18 \arrow[ur] \arrow[dr] & \\
c_0 \approx 0.10 \arrow[ur] \arrow[dr] & & c_{du} = 0.08 \\
 & c_d \approx 0.036 \arrow[ur] \arrow[dr] & \\
 & & c_{dd} = 0
\end{tikzcd}$$

## Two-period call option example

- Want also illustration of replicating strategy.
- Solution derived backwards, but now look forwards.
- Consider first $(\Delta_0, B_0)$ at $t = 0$:

$$\Delta_0 = \frac{c_u - c_d}{(u-d)S_0} = \frac{\frac{8}{55}}{0.2} = \frac{8}{11},$$

$$B_0 = \frac{uc_d - dc_u}{(u-d)e^r} = \frac{1.2\frac{2}{55} - \frac{2}{11}}{0.2 \cdot 1.1} = -\frac{76}{121}.$$

- Observe $\Delta_0 S_0 + B_0 = \frac{88-76}{121} = \frac{12}{121} = c_0$.
- Consider what happens at $t = 1$ if $S_1 = u \cdot S_0$.
- Value of $(\Delta_0, B_0)$ pf. will then be

$$1.2\frac{8}{11} - 1.1\frac{76}{121} = \frac{2}{11}.$$

- Will show this is exactly needed to buy $(\Delta_u, B_u)$:

$$\Delta_u = \frac{c_{uu} - c_{du}}{(u-d)uS_0} = \frac{0.24}{0.2 \cdot 1.2} = 1,$$

$$B_u = \frac{uc_{du} - dc_{uu}}{(u-d)e^r} = \frac{1.2 \cdot 0.08 - 0.32}{0.2\frac{11}{10}} = -\frac{56}{55}.$$

- This implies $\Delta_u \cdot u \cdot S_0 + B_u = 1.2 - \frac{56}{55} = \frac{2}{11}$.
- Selling $(\Delta_0, B_0)$ pf. gives exactly what is needed.
- Can show (yourself!) the same for $S_1 = d \cdot S_0$.

## Interpretation of $p \equiv (e^r - d)/(u - d)$

- So far, $p$ is introduced purely to simplify.
- Call option value derived from absence-of-arbitrage proofs.
- Formula simplified when $p$ introduced.
- But formula looks suspiciously like expected present value:

$$C_0 = \frac{pC_u + (1-p)C_d}{e^r}.$$

- Interpretation requires $p$ interpreted as probability.
- The requirement $p \in [0, 1]$ is OK since $d \leq e^r \leq u$.
- But actual probability is $p^*$.
- $p^*$ does not appear in $C_0$ formula.
- In particular, $p^*$ does not appear in definition of $p$.
- Nevertheless interpret $p$ as probability.
- $p$ appears as probability in alternative, imagined world.
- That world has risk neutral individuals.

## $p$ as probability in risk neutral world

- Risk neutral individuals care only about expectations.
- In equilibrium in "risk neutral world," all assets must have same expected rates of return.
- No one would hold those with smaller expected rates of return.
- Consider *thought experiment*:
- Keep specification of our binomial model, except for $p^*$.
- In particular, keep $u, d, r, S_0$ unchanged. These determine option value, so this value is also unchanged.
- What probability would we need to make $E(S_1/S_0) = e^r$?
- Call answer $p_x$:

$$p_x \cdot u \cdot S_0 + (1 - p_x) \cdot d \cdot S_0 = e^r \cdot S_0$$

  has solution $p_x = (e^r - d)/(u - d) \equiv p$.
- Conclude: $p$ is the value needed for the probability in our binomial model of the share price, in order for bonds and shares to have the same expected rate of return.
- $p$ is thus "probability in risk neutral world" for $u$.

## Risk neutral interpretation, contd.

- Clearly $p \neq p^*$ for most shares (except for which $\beta$?).
- When going to the risk neutral world:
- Nothing in a-o-arbitrage option value is changed.
- Option value did not rely on the value of $p^*$.
- Composition of replicating portfolio unchanged.
- Thus option value still $C_0 = e^{-r}[pC_u + (1-p)C_d]$.
- But can also be seen from requirement $E(C_1/C_0) = e^r$:

$$pC_u + (1-p)C_d = e^r C_0$$

  has solution $C_0 = e^{-r}[pC_u + (1-p)C_d]$.
- Thus: Could derive $C_0$ as expected present value.
- But then: Need to use "artificial" probability $p$.
- Sometimes called "risk-neutral" expected present value.

## When is this insight useful?

- For European call options no need for "risk-neutral" valuation.
- But the principle has more widespread application.
- Sometimes difficult to find formulae for replicating pf.
- May nevertheless know replication possible in principle.
- Expected (using $p$) present values may be easier to calculate.

## Approximating real world with binomial model

- Two unrealistic features of binomial model:
  - In reality shares can change their values to any positive number, not only those specified in tree.
  - In reality trade in shares and other securities can happen all the time, not at given time intervals.
- Nevertheless useful model, pedagogically, numerically (ch. 19).
- Specify variables in order to approximate reality.
- Let interval time length $h \equiv T/n$ go to zero.
- $T$ measures calendar time until expiration. Fixed.
- $n$ is number of periods (intervals) we divide $T$ into.
- Let $n \to \infty, \quad h \to 0$.
- $S_T$ becomes product of many independent variables, e.g.,

$$S_T = d \cdot d \cdot u \cdot d \cdot u \cdot u \cdot d \cdot u \cdot u \cdot S_0.$$

- Better to work with $\ln(S_T/S_0)$, rewritten

$$\ln\left(\frac{S_T}{S_0}\right) = j\ln(u) + (n-j)\ln(d).$$

  where $j$ is a binomial random variable.
- Central limit theorem: When $n \to \infty$, the expression $\ln(S_T/S_0)$ approaches a normally distributed random variable.

## Normal and lognormal distributions

When $\ln(S_T/S_0) = \ln(S_T) - \ln(S_0)$ is normally distributed:

- $\ln(S_T)$ also normally distr., since $\ln(S_0)$ is a constant.
- $S_T/S_0$ and $S_T$ are *lognormally* distributed.
- $\ln(S_T)$ can take any (real) value, positive or negative.
- $S_T$ can take any *positive* value.
- Graphs show normal and lognormal distributions:

# The transition to the continuous-time model

- Any positive $S_T$ value will have positive probability.
- But over a short period of time (e.g., one week), large (and very small) $S_T/S_0$ will have very low probability.
- Black and Scholes wrote model in continuous time directly.
- To exploit arbitrage: Must adjust replicating pf. all the time.
- Relies on (literally) *no* transaction costs.
- Sufficient that some people have no transaction costs: They will use arbitrage opportunity if available.
- Small fixed costs of "each transaction" destroys model.
- New unrealistic feature introduced (?)
- Consistent, consequence of assuming no transaction costs.

# Mathematics of transition to continuous time

- Will need to make $u, d, r, p^*$ functions of $h$ (or $n$).
- If not: Model would "explode" when $n \to \infty$.
- Specifically, $e^{rn} \to \infty$.
- Also, if $u > d$, then $E[\ln(S_T/S_0)] \to \infty$.
- Instead, start with some reasonable values for
$$e^{rT},$$
$$\mu T \equiv E[\ln(S_T/S_0)],$$
and
$$\sigma^2 T \equiv \text{var}[\ln(S_T/S_0)],$$
for instance observed from empirical data.
- Then choose $u, d, \hat{r}, p^*$ for binomial model as functions of $h$ such that as $h \to 0$, the binomial model's $e^{\hat{r}n}$, $E[\ln(S_T/S_0)]$, and $\text{var}[\ln(S_T/S_0)]$ approach the "starting" values (above).
- Easiest:
$$e^{\hat{r}n} = e^{rT} \Rightarrow \hat{r} = rT/n = rh.$$

## $u, d, p^*$ as functions of $h$

Remember

$$\ln\left(\frac{S_T}{S_0}\right) = j\ln\left(\frac{u}{d}\right) + n\ln(d),$$

with $j$ binomial, the number of $u$ outcomes in $n$ independent draws, each with probability $p^*$.

Let $\hat{\mu}, \hat{\sigma}$ belong in binomial model:

$$\hat{\mu}n \equiv E[\ln(S_T/S_0)] = [p^*\ln(u/d) + \ln(d)]n,$$

$$\hat{\sigma}^2 n \equiv \text{var}[\ln(S_T/S_0)] = p^*(1-p^*)\left[\ln(u/d)\right]^2 n.$$

- Want $h \to 0$ to imply both $\hat{\mu}n \to \mu t$ and $\hat{\sigma}^2 n \to \sigma^2 t$.
- Free to choose $u, d, p^*$ to obtain two goals.
- Many ways to achieve this, one degree of freedom.
- Choose that one which makes $S$ a continuous function of time.
- No jumps in time path.
- Necessary in order to be able to adjust replicating portfolio.

## $u, d, p^*$ as functions of $h$, contd.

Let $u = e^{\sigma\sqrt{h}}$, $d = e^{-\sigma\sqrt{h}}$, $p^* = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{h}$. Will show these choices work:

$$\hat{\mu}n = \left[\left(\frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{h}\right) \cdot 2\sigma\sqrt{h} - \sigma\sqrt{h}\right] n$$

$$= \left(\sigma\sqrt{\frac{t}{n}} + \mu\frac{t}{n} - \sigma\sqrt{\frac{t}{n}}\right) n = \mu t,$$

$$\hat{\sigma}^2 n = \left(\frac{1}{2}\frac{\mu}{\sigma}\sqrt{h}\right)\left(\frac{1}{2} - \frac{1}{2}\frac{\mu}{\sigma}\sqrt{h}\right) \cdot 4 \cdot \sigma^2 hn$$

$$= \left(\frac{1}{4} - \frac{1}{4}\frac{\mu^2}{\sigma^2}\frac{t}{n}\right) \cdot 4\sigma^2 t \to \sigma^2 t \text{ when } n \to \infty.$$

Observe that our choices make $u$ and $d$ independent of the value of $\mu$. Thus the option value, which depends on $u$ and $d$, but not of $p^*$, will be independent of $\mu$.

**Example of convergence, $h \to 0$**

Let $T = 2$ and assume that we want convergence to

$$2\mu = E[\ln(S_T/S_0)] = 0.08926, \quad 2\sigma^2 = \text{var}[\ln(S_T/S_0)] = 0.09959$$

so that $\mu = 0.04463$ and $\sigma = 0.2231$.

Choose

$$u = e^{\sigma\sqrt{h}},\ d = e^{-\sigma\sqrt{h}} = \frac{1}{u},\ p^* = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{h}.$$

Will show what the numbers look like when $n = 2$ and when $n = 4$.

For $n = 2$ (i.e., $h = T/n = 1$) we find

$$u = e^{\sigma} = 1.25,\ d = \frac{1}{u} = 0.8,\ p^* = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma} = 0.6,$$

which yields, in the binomial model,

$$\hat{\mu}n = E\left[\ln\left(\frac{S_T}{S_0}\right)\right] = 0.08926, \quad \hat{\sigma}^2 n = \text{var}\left[\ln\left(\frac{S_T}{S_0}\right)\right] = 0.09560.$$

For $n = 4$ (i.e., $h = T/n = 1/2$) we find

$$u = e^{\sigma\sqrt{0.5}} = 1.1709,\ d = \frac{1}{u} = 0.8540,\ p^* = \frac{1}{2}+\frac{1}{2}\frac{\mu}{\sigma}\sqrt{0.5} = 0.5707,$$

which yields, in the binomial model,

$$\hat{\mu}n = E\left[\ln\left(\frac{S_T}{S_0}\right)\right] = 0.08926, \quad \hat{\sigma}^2 n = \text{var}\left[\ln\left(\frac{S_T}{S_0}\right)\right] = 0.09759.$$

This indicates that as $h \to 0$, we have $u \to 1,\ d \to 1$, $p^* \to 0.5$, and $\hat{\sigma}^2 n \to \sigma^2 t$.