Stochastic dominance

- Two criteria for making decisions without knowing shape of $U()$.
- May be important for delegation, for research, for prediction: Situations in which you are not able to point out exactly which U function is the right one to use.
- These two criteria (see below) work only for some types of comparisons. For other comparisons, these decision criteria are inconclusive.
- When you have many (more than two) alternatives, it will often turn out that neither of the two dominance criteria give you an answer to which alternative is the best. But one of them (or both) can nevertheless be useful for narrowing down choices by excluding dominated alternatives.

First-order stochastic dominance

A random variable $\tilde{X}_A$ *first-order stochastically dominates* another random variable $\tilde{X}_B$ if every vN-M expected utility maximizer prefers $\tilde{X}_A$ to $\tilde{X}_B$.

Second-order stochastic dominance

A random variable $\tilde{X}_A$ *second-order stochastically dominates* another random variable $\tilde{X}_B$ if every *risk-averse* vN-M expected utility maximizer prefers $\tilde{X}_A$ to $\tilde{X}_B$.

First-order stochastic dominance, FSD

(Let the cumulative distribution functions be $F_A(x) \equiv \Pr(\tilde{X}_A \leq x)$ and $F_B(x) \equiv \Pr(\tilde{X}_B \leq x)$.)

Possible to show that "$\tilde{X}_A \succ \tilde{X}_B$ by all" is equivalent to the following, which is one possible definition of first-order s.d.:

$$F_A(w) \leq F_B(w) \text{ for all } w,$$

and

$$F_A(w_i) < F_B(w_i) \text{ for some } w_i.$$

For any level of wealth w, the probability that $\tilde{X}_A$ ends up below that level is less than the probability that $\tilde{X}_B$ ends up below it.

Second-order stochastic dominance, SSD

Possible to show that "$\tilde{X}_A \succ \tilde{X}_B$ by all risk averters" is equivalent to the following, which is one possible definition of second-order s.d.:

$$\int_{-\infty}^{w_i} F_A(w)dw \leq \int_{-\infty}^{w_i} F_B(w)dw \text{ for all } w_i,$$

and

$$F_A(w_i) \neq F_B(w_i) \text{ for some } w_i.$$

One distribution is more dispersed ("more uncertain") than the other. If we restrict attention to variables $\tilde{X}_A$ and $\tilde{X}_B$ with the same expected value, Theorem 4.4 in D&D states that SSD is equivalent to: $\tilde{X}_B$ can be written as $\tilde{X}_A + \tilde{z}$, where the difference $\tilde{z}$ is some random noise.

Risk aversion and simple portfolio problem

(Chapter 5 in Danthine and Donaldson.)

Simple portfolio problem, one risky, one risk free asset. Total investment is Y_0, a part of this, a, is invested in risky asset with rate of return $\tilde{r}$, while $Y_0 - a$ is invested at risk free rate r_f. Expected utility becomes a function of a, which the investor wants to maximize by choosing a:

$$W(a) \equiv E\{U[\tilde{Y}_1]\} \equiv E\{U[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}, \qquad (1)$$

based on $\tilde{Y}_1 = (Y_0 - a)(1 + r_f) + a(1 + \tilde{r})$.

Solution of course depends on investor's U function. Assuming $U'' < 0$ and interior solutions $(0 \leq a_0 \leq Y_0)$ we can show:

- Optimal a strictly positive if and only if $E(\tilde{r}) > r_f$.
- When the optimal a is strictly positive:
  - Optimal a independent of Y_0 for CARA, increasing in Y_0 for DARA, decreasing in Y_0 for IARA.
  - (CARA means Constant absolute risk aversion, DARA means Decreasing ARA, IARA means Increasing ARA.)
  - Optimal a/Y_0 independent of Y_0 for CRRA, increasing in Y_0 for DRRA, decreasing in Y_0 for IRRA.
  - (CRRA, DRRA, IRRA refer to relative risk aversion instead of absolute.)

This gives a better understanding of what it means to have, e.g., decreasing absolute risk aversion.

First-order condition for simple portfolio problem

To find f.o.c. of maximization problem (1), need take partial derivative of expectation of something with respect to a deterministic variable. Straight forward when $\tilde{r}$ has discrete probability distribution, with π_θ the probability of outcome r_θ. Then $W(a) =$

$$E\{U[Y_0(1+r_f)+a(\tilde{r}-r_f)]\} = \sum_\theta \pi_\theta U[Y_0(1+r_f)+a(r_\theta-r_f)],$$

and the f.o.c. with respect to a is

$$\begin{aligned} W'(a) &= \sum_\theta \pi_\theta U'[Y_0(1+r_f)+a(r_\theta-r_f)](r_\theta-r_f) \\ &= E\{U'[Y_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)\} = 0. \qquad (2) \end{aligned}$$

The final equation above, (2), is also f.o.c. when distribution continuous, cf. Leibniz' formula (see Sydsæter et al): The derivative of a definite integral (with respect to some variable other than the integration variable) is equal to the definite integral of the derivative of the integrand.

Observe that in (2) there is the expectation of a product, and that the two factors $U'[Y_0(1+r_f)+a(\tilde{r}-r_f)]$ and $(\tilde{r}-r_f)$ are not stochastically independent, since they depend on the same stochastic variable $\tilde{r}$. Thus this is not equal to the product of the expectations.

Prove: Invest in risky asset if and only if $E(\tilde{r}) > r_f$

Repeat: $W(a) \equiv E\{U[Y_0(1+r_f) + a(\tilde{r} - r_f)]\}$.

Consider $W''(a) = E\{U''[Y_0(1+r_f) + a(\tilde{r} - r_f)](\tilde{r} - r_f)^2\}$. The function $W(a)$ will be concave since U is concave. Consider now the first derivative when $a = 0$:

$$W'(0) = E\{U'([Y_0(1+r_f)](\tilde{r}-r_f)\} = U'[Y_0(1+r_f)]E(\tilde{r}-r_f). \quad (3)$$

We find:

- If $E(\tilde{r}) > r_f$, then (3) is positive, which means that E(U)=W will be increased by increasing a from $a = 0$. The optimal a is thus strictly positive.
- If $E(\tilde{r}) < r_f$, then (3) is negative, which means that E(U)=W will be increased by decreasing a from $a = 0$. The optimal a is thus strictly negative.
- If $E(\tilde{r}) = r_f$, then (3) is zero, which means that the f.o.c. is satisfied at $a = 0$. The optimal a is zero.

Of course, $a < 0$ means short-selling the risky asset, which may or may not be possible and legal.

The connection between a, Y_0, and $R_A(Y_1)$

(Theorem 5.4 in Danthine and Donaldson)

The result to prove is that the optimal a is independent of Y_0 for CARA, increasing in Y_0 for DARA, decreasing in Y_0 for IARA (assuming all the time that optimal $a > 0$).

Total differentiation of first-order condition with respect to a and Y_0:

$$E\{U''[Y_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)^2\}da$$

$$+ E\{U''([Y_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)(1+r_f)\}dY_0 = 0$$

gives

$$\frac{da}{dY_0} = -\frac{E\{U''[Y_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)\}(1+r_f)}{E\{U''[Y_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)^2\}}$$

Denominator is always negative. Considering also the minus sign in front, we see that the whole expression has the same sign as the numerator. Will show this is positive for DARA. Similar proof that it is zero for CARA and negative for IARA.

da/dY_0 under Decreasing absolute risk aversion

DARA means that $R_A(Y) \equiv -U''(Y)/U'(Y)$ is a decreasing function, i.e., $R'_A(Y) < 0$ for all positive Y.

Consider first outcomes $r_\theta > r_f$. DARA implies $R_A(Y_0(1+r_f) + a(r_\theta - r_f)) < R_A(Y_0(1+r_f))$, which can be rewritten

$$U''[Y_0(1+r_f)+a(r_\theta-r_f)] > -R_A(Y_0(1+r_f))U'[Y_0(1+r_f)+a(r_\theta-r_f)].$$

Multiply by the positive $(r_\theta - r_f)$ on both sides to get

$$U''[Y_0(1+r_f) + a(r_\theta - r_f)](r_\theta - r_f)$$

$$> -R_A(Y_0(1+r_f))U'[Y_0(1+r_f) + a(r_\theta - r_f)](r_\theta - r_f). \quad (4)$$

Consider next outcomes $r_\theta < r_f$. DARA implies $R_A(Y_0(1+r_f) + a(r_\theta - r_f)) > R_A(Y_0(1+r_f))$, rewritten

$$U''[Y_0(1+r_f)+a(r_\theta-r_f)] < -R_A(Y_0(1+r_f))U'[Y_0(1+r_f)+a(r_\theta-r_f)].$$

Multiply by the negative $(r_\theta - r_f)$ on both sides to get

$$U''[Y_0(1+r_f) + a(r_\theta - r_f)](r_\theta - r_f)$$

$$> -R_A(Y_0(1+r_f))U'[Y_0(1+r_f) + a(r_\theta - r_f)](r_\theta - r_f). \quad (5)$$

Clearly, (4) and (5) are the same inequality. This therefore holds for both $r_\theta > r_f$ and $r_\theta < r_f$. Then it also holds for the expectations of the LHS and the RHS: $E\{U''[Y_0(1+r_f) + a(\tilde{r} - r_f)](\tilde{r} - r_f)\}$

$$> -R_A(Y_0(1+r_f))E\{U'[Y_0(1+r_f) + a(\tilde{r} - r_f)](\tilde{r} - r_f)\}$$

which is zero by the first-order condition, q.e.d.

Risk aversion and saving

(Sect. 5.6, D&D.) How does saving depend on riskiness of return? Rate of return is $\tilde{r}$, (gross) return is $\tilde{R} \equiv 1 + \tilde{r}$. Consider choice of saving, s, when probability distribution of $\tilde{R}$ is taken as given:

$$\max_{s \in R_+} E[U(Y_0 - s) + \delta U(s\tilde{R})]$$

where Y_0 is a given wealth, δ is (time) discount factor for utility.

Rewrite,

$$\max_{s \in R_+} U(Y_0 - s) + \delta E[U(s\tilde{R})],$$

first-order condition,

$$-U'(Y_0 - s) + \delta E[U'(s\tilde{R})\tilde{R}] = 0.$$

Risk aversion and saving, contd.

- Savings decision well known topic in microec. without risk
- Typical questions: Depedence of s on Y_0 and on $E(\tilde{R})$
- Focus here: How does saving depend on riskiness of $\tilde{R}$?
- Consider mean-preserving spread: Keep $E(\tilde{R})$ fixed
- Assuming risk aversion, answer is not obvious:
  - $\tilde{R}$ more risky means saving is less attractive, $\Rightarrow$ save less
  - $\tilde{R}$ more risky means probability of low $\tilde{R}$ higher, willing to give up more of today's consumption to avoid low consumption levels next period, $\Rightarrow$ save more
- Need to look carefully at first-order condition

$$U'(Y_0 - s) = \delta E[U'(s\tilde{R})\tilde{R}]$$

- What happens to right-hand side as $\tilde{R}$ becomes more risky?
- Cannot conclude in general, but for some conditions on U
- (Jensen's inequality:) Depends on concavity of $g(R) \equiv U'(sR)R$
- If, e.g., g is concave:
  - May compare risk with no risk: $E[g(\tilde{R})] < g[E(\tilde{R})]$
  - But also some risk with more risk, cf. Theorem 5.7 in D&D.

Risk aversion and saving, contd.

$$\max_{s \in R_+} E[U(Y_0 - s) + \delta U(s\tilde{R})]$$

(assuming, all the time here, $U' > 0$ and risk aversion, $U'' < 0$)

- When $\tilde{R}_B = \tilde{R}_A + \tilde{\varepsilon}$, $E(\tilde{R}_B) = E(\tilde{R}_A)$, will show:
  - If $R_R'(Y) \leq 0$ and $R_R(Y) > 1$, then $s_A < s_B$.
  - If $R_R'(Y) \geq 0$ and $R_R(Y) < 1$, then $s_A > s_B$.
- First condition on each line concerns IRRA vs. DRRA, but both contain CRRA.
- Second condition on each line concerns magnitude of R_R (also called RRA): Higher risk aversion implies save more when risk is high. Lower risk aversion (than $R_R = 1$) implies save less when risk is high. But none of these claims hold generally; need the respective conditions on sign of R_R'.
- Interpretation: When risk aversion is high, it is very important to avoid the bad outcomes in the future, thus more is saved when the risk is increased.
- Reminder: This does *not* mean that a highly risk averse person puts more money into any asset the more risky the asset is. In this model, the portfolio choice is assumed away. If there had been a risk free asset as well, the more risk averse would save in that asset instead.

Risk aversion and saving, contd.

Proof for the first case, $R'_R(Y) \leq 0$ and $R_R(Y) > 1$:

Use $g'(R) = U''(sR)sR + U'(sR)$ and $g''(R) = U'''(sR)s^2R + 2U''(sR)s$. For g to be convex, need $U'''(sR)sR + 2U''(sR) > 0$. To prove that this holds, use

$$R'_R(Y) = \frac{[-U'''(Y)Y - U''(Y)]U'(Y) - [-U''(Y)Y]U''(Y)}{[U'(Y)]^2},$$

which implies that $R'_R(Y)$ has the same sign as

$$-U'''(Y)Y - U''(Y) - [-U''(Y)Y]U''(Y)/U'(Y)$$

$$= -U'''(Y)Y - U''(Y)[1 + R_R(Y)].$$

When $R'_R(Y) < 0$, and $R_R(Y) > 1$, this means that

$$0 < U'''(Y)Y + U''(Y)[1 + R_R(Y)] < U'''(Y)Y + U''(Y) \cdot 2.$$

Since this holds for all Y, in particular for $Y = sR$, we find

$$U'''(sR)sR + 2U''(sR) > 0,$$

and g is thus convex.

Mean-variance versus vN-M expected utility

- Chapters 6 and 7 of D & D relies on the “mean-variance” assumption.
- Individuals are assumed to care about only the expected value (“mean”) and variance of their future risky consumption possibilities.
- In general those who maximize $E[U(\tilde{W})]$ care about the whole distribution of $\tilde{W}$.
- Will care about only mean and variance if those two characterize the whole distribution.
- Will alternatively care about only mean and variance if $U()$ is a quadratic function.

The third way to underpin mean-var assumption

- Perhaps things are so complicated that people resort to just considering mean and variance. (Whether they are vN-M people or not.)

Mean-var preferences due to distribution

- Assume that choices are always between random variables with one particular type ("class") of probability distribution.
- Could be, e.g., choice only between binomially distributed variables. (There are different binomial distributions, summarized in three parameters which uniquely define each one of them.)
- Or, e.g., only between variables with a chi-square distribution. Or variables with normal distribution. Or variables with a lognormal distribution.
- Some of these distributions, such as the normal distribution and the lognormal distribution, are characterized completely by two parameters, the mean and the variance.
- If all possible choices belong to the same class, then the choice can be made on the basis of the parameters for each of the distributions.
- Example: Would you prefer a normally distributed wealth with mean 1000 and variance 40000 or another normally distributed wealth with mean 500 and variance 10000?

- If mean and variance characterize each alternative completely, then all one cares about is mean and variance.
- Most convenient: Normal distribution, since sums (and more generally, any linear combinations) of normally distributed variables are also normal. Most opportunity sets consist of alternative linear combinations of variables.
- Problem: Positive probability for negative outcomes. Share prices are never negative.

Mean-var preferences due to quadratic U

Assume

$$U(w) \equiv cw^2 + bw + a$$

where $b > 0, c < 0$, and a are constants. With this U function:

$$\begin{aligned} E[U(\tilde{W})] &= cE(\tilde{W}^2) + bE(\tilde{W}) + a \\ &= c\{E(\tilde{W}^2) - [E(\tilde{W})]^2\} + c[E(\tilde{W})]^2 + bE(\tilde{W}) + a \\ &= c\,\mathrm{var}(\tilde{W}) + c[E(\tilde{W})]^2 + bE(\tilde{W}) + a, \end{aligned}$$

which is a function only of mean and variance of $\tilde{W}$.

Problem: U function is decreasing for large values of W. Must choose c and b such that those large values have zero probability.

Another problem: Increasing (absolute) risk aversion.

Indifference curves in mean-stddev diagrams

- If mean and variance are sufficient to determine choices, then mean and $\sqrt{\text{variance}}$ are also sufficient.
- More practical to work with mean (μ) and standard deviation (σ) diagrams.
- Common to put standard deviation on horizontal axis.
- Will show that indifference curves are increasing and convex in (σ, μ) diagrams; also slope $\to 0$ as $\sigma \to 0^+$.
- Consider normal distribution and quadratic U separately.
- Indifference curves are contour curves of $E[U(\tilde{W})]$.
- Total differentiation:

$$0 = dE[U(\tilde{W})] = \frac{\partial E[U(\tilde{W})]}{\partial \sigma} d\sigma + \frac{\partial E[U(\tilde{W})]}{\partial \mu} d\mu.$$

Indifference curves from quadratic U

Assume $W < -b/(2c)$ with certainty in order to have $U'(W) > 0$.

$$E[U(\tilde{W})] = c\sigma^2 + c\mu^2 + b\mu + a.$$

First-order derivatives:

$$\frac{\partial E[U(\tilde{W})]}{\partial \sigma} = 2c\sigma < 0, \ \frac{\partial E[U(\tilde{W})]}{\partial \mu} = 2c\mu + b > 0,$$

Thus the slope of the indifference curves,

$$\left.\frac{d\mu}{d\sigma}\right|_{E[U(\tilde{W})] \text{ const.}} = -\frac{\frac{\partial E[U(\tilde{W})]}{\partial \sigma}}{\frac{\partial E[U(\tilde{W})]}{\partial \mu}} = \frac{-2c\sigma}{2c\mu + b},$$

is positive, and approaches 0 as $\sigma \to 0^+$.

Second-order:

$$\frac{\partial^2 E[U(\tilde{W})]}{\partial \sigma^2} = 2c < 0, \ \frac{\partial^2 E[U(\tilde{W})]}{\partial \mu^2} = 2c < 0, \ \frac{\partial^2 E[U(\tilde{W})]}{\partial \mu \partial \sigma} = 0.$$

The function is concave, thus it is also quasi-concave.

Indifference curves from normally distributed $\tilde{W}$

Let $f(\varepsilon) \equiv (1/\sqrt{2\pi})e^{-\varepsilon^2/2}$, the std. normal density function. Let $W = \mu + \sigma\varepsilon$, so that $\tilde{W}$ is $N(\mu, \sigma^2)$.

Define expected utility as a function

$$E[U(\tilde{W})] = V(\mu, \sigma) = \int_{-\infty}^{\infty} U(\mu + \sigma\varepsilon) f(\varepsilon) d\varepsilon.$$

Slope of indifference curves:

$$-\frac{\frac{\partial V}{\partial \sigma}}{\frac{\partial V}{\partial \mu}} = \frac{-\int_{-\infty}^{\infty} U'(\mu + \sigma\varepsilon)\varepsilon f(\varepsilon) d\varepsilon}{\int_{-\infty}^{\infty} U'(\mu + \sigma\varepsilon) f(\varepsilon) d\varepsilon}.$$

Denominator always positive. Will show that integral in numerator is negative, so minus sign makes the whole fraction positive.

Integration by parts: Observe $f'(\varepsilon) = -\varepsilon f(\varepsilon)$. Thus:

$$\int U'(\mu + \sigma\varepsilon)\varepsilon f(\varepsilon) d\varepsilon = -U'(\mu + \sigma\varepsilon) f(\varepsilon) + \int U''(\mu + \sigma\varepsilon)\sigma f(\varepsilon) d\varepsilon.$$

First term on RHS vanishes in limit when $\varepsilon \to \pm\infty$, so that

$$\int_{-\infty}^{\infty} U'(\mu + \sigma\varepsilon)\varepsilon f(\varepsilon) d\varepsilon = \int_{-\infty}^{\infty} U''(\mu + \sigma\varepsilon)\sigma f(\varepsilon) d\varepsilon < 0.$$

Another important observation:

$$\lim_{\sigma \to 0^+} \frac{d\mu}{d\sigma} = \frac{-U'(\mu) \int_{-\infty}^{\infty} \varepsilon f(\varepsilon) d\varepsilon}{U'(\mu) \int_{-\infty}^{\infty} f(\varepsilon) d\varepsilon} = 0.$$

To show concavity of $V()$:

$$
\begin{aligned}
&\lambda V(\mu_1, \sigma_1) + (1-\lambda)V(\mu_2, \sigma_2) \\
&= \int_{-\infty}^{\infty} [\lambda U(\mu_1 + \sigma_1\varepsilon) + (1-\lambda)U(\mu_2 + \sigma_2\varepsilon)] f(\varepsilon) d\varepsilon \\
&< \int_{-\infty}^{\infty} U(\lambda\mu_1 + \lambda\sigma_1\varepsilon + (1-\lambda)\mu_2 + (1-\lambda)\sigma_2\varepsilon) f(\varepsilon) d\varepsilon \\
&= V(\lambda\mu_1 + (1-\lambda)\mu_2, \lambda\sigma_1 + (1-\lambda)\sigma_2).
\end{aligned}
$$

The function is concave, thus it is also quasi-concave.

Mean-variance portfolio choice

- One individual, mean-var preferences, a given W_0 to invest at $t = 0$
- Regards probability distribution of future ($t = 1$) values of securities as exogenous; values at $t = 1$ include payouts like dividends, interest
- Today also: Regards security prices at $t = 0$ as exogenous
- Later: Include this individual in equilibrium model of competitive security market at $t = 0$

Notation: Investment of W_0 in n securities:

$$W_0 = \sum_{j=1}^{n} p_{j0} X_j = \sum_{j=1}^{n} W_{j0}$$

Value of this one period later:

$$\tilde{W} = \sum_{j=1}^{n} \tilde{p}_{j1} X_j = \sum_{j=1}^{n} \tilde{W}_j = \sum_{j=1}^{n} p_{j0} \frac{\tilde{p}_{j1}}{p_{j0}} X_j$$

$$= \sum_{j=1}^{n} p_{j0}(1 + \tilde{r}_j) X_j = \sum_{j=1}^{n} W_{j0}(1 + \tilde{r}_j)$$

$$= W_0 \sum_{j=1}^{n} \frac{W_{j0}}{W_0}(1 + \tilde{r}_j) = W_0 \sum_{j=1}^{n} w_j (1 + \tilde{r}_j) = W_0(1 + \tilde{r}_p),$$

where the w_j's, known as portfolio weights, add up to unity.

$\tilde{r}_j = \frac{\tilde{P}_{j1}}{P_{j0}} - 1$ is *rate of return* on asset j

Mean-var preferences for rates of return

$$\tilde{W} = W_0 \sum_{j=1}^{n} w_j(1+\tilde{r}_j) = W_0 \left(1 + \sum_{j=1}^{n} w_j \tilde{r}_j\right) = W_0(1+\tilde{r}_p)$$

- $\tilde{r}_p$ is rate of return for investor's portfolio
- If each investor's W_0 fixed, then preferences well defined over $\tilde{r}_p$, may forget about W_0 for now
- Let $\mu_p \equiv E(\tilde{r}_p)$ and $\sigma_p^2 \equiv \operatorname{var}(\tilde{r}_p)$; then

$$E(\tilde{W}) = W_0(1+E(\tilde{r}_p)) = W_0(1+\mu_p),$$
$$\operatorname{var}\tilde{W} = W_0^2 \operatorname{var}(\tilde{r}_p),$$
$$\sqrt{\operatorname{var}(\tilde{W})} = W_0\sqrt{\operatorname{var}(\tilde{r}_p)} = W_0\sigma_p$$

Increasing, convex indifference curves in $(\sqrt{\operatorname{var}(\tilde{W})}, E(\tilde{W}))$ diagram imply increasing, convex indifference curves in (σ_p, μ_p) diagram

But: A change in W_0 will in general change the shape of the latter kind of curves ("wealth effect")