Valuation of options *before* expiration

- Need to distinguish between American and European options.
- Consider European options with time t until expiration.
- Value now of receiving c_T at expiration?
- (Value now of receiving p_T at expiration? Later.)
- Have candidate model already: Use CAPM?
- Problematic: Non-linear functions of S_T.
- Difficult to calculate $E(c_T)$ and $\text{cov}(c_T, r_M)$.
- Instead: Theory especially developed for options.
- (But turns out to have other applications as well.)
- "Valuation of derivative assets."
- Value of one asset as function of value of another.
- Will find $c(S, \ldots)$ and $p(S, \ldots)$.
- Other variables as arguments besides S.
- Want those other variables to be *observables.*

Net value diagrams (Hull, figs. 9-1–9-4, 11-1–11-12)[1]

- Value at expiration *minus* purchase cost.
- S_T on horizontal axsis.
- Example: $c_T - c$, buying a call option.
- Resembles gross value, c_T, diagram.
- But removed vertically by subtracting c, today's price.
- These diagrams only approximately true:
- No present-value correction for time lag $-c$ to c_T.
- There isn't one exact relationship between c and c_T.
- That exact relationship depends on other variables.
- Next example: $c - c_T$, selling a call option.
- Observe: Selling and buying cancel out for each S_T.
- Options redistribute risks (only). Zero-sum.
- Examples on this and next two pages: Carefully chosen.

[1] 7th ed.: figs. 8-1–8-4, 10-1–10-12

Net value diagrams, contd.

- Buy a share and buy two put options with $S = K$.
- Diagrams show $S_T - S$, $2p_T - 2p$, $S_T - S + 2p_T - 2p$.
- Good idea if you believe S_T will be different from S (and K), but you do not know direction.

Net value diagrams, contd.

- Buy put option with $K = S$, plus one share.
- $p_T - p + S_T - S$.
- Resembles value of call option.
- Will soon show exact relationship to call option.

Determinants of option value (informally)

Six candidates for explanatory variables for c and p:

- S, today's share price. Higher S means market expects higher S_T, implies higher c (because higher c_T), lower p (lower p_T).
- K, the striking price. Higher K means lower c (because lower c_T), higher p (higher p_T).
- Uncertainty. Higher uncertainty implies both higher c *and* higher p, because option owner gains from extreme outcomes in one direction, while being protected in opposite direction. (Remark: This is total risk in S_T, not β from CAPM.)
- Interest rate. Higher interest rate implies present value of K is reduced, increasing c, decreasing p.
- Time until expiration. Two effects (for a fixed uncertainty per unit of time): Longer time implies increased uncertainty about S_T, and lower present value of K. Both give higher c, while effects on p go in opposite directions.
- Dividends. If share pays dividends before expiration, this reduces expected S_T (for a given S, since S is claim to both dividend and S_T). Option only linked to S_T, thus lower c, higher p.

Later: Precise formula for $c(S, K, \sigma, r, t)$ when $D = 0$.

Missing from the list: $E(S_T)$. Main achievement!

Put-call parity

Exact relationship between call and put values.

- Assume underlying share with certainty pays no dividends between now and expiration date of options.
- Let t = time until expiration date.
- Consider European options with same K, t.
- Consider following set of four transactions:

		At expiration	
	Now	If $S_T \leq K$	If $S_T > K$
Sell call option	c	0	$K - S_T$
Buy put option	$-p$	$K - S_T$	0
Buy share	$-S$	S_T	S_T
Borrow (risk free)	Ke^{-rt}	$-K$	$-K$
Total	$c - p - S + Ke^{-rt}$	0	0

Must have $c = p + S - Ke^{-rt}$, if not, riskless arbitrage.

- To exploit arbitrage if, e.g., $c > p + S - Ke^{-rt}$:
- "Buy cheaper, sell more expensive."
- Sell (i.e., write) call option.
- Buy put option and share.
- Borrow Ke^{-rt}.
- Receive $c - p - S + Ke^{-rt} > 0$ now.
- At expiration: Net outlay zero whatever S_T is.

Put-call parity allows us to concentrate on (e.g.) calls.

Allow for uncertain dividends

- Share may pay dividends before expiration of option.
- These drain share value, do not accrue to call option.
- In Norway dividends paid once a year, in U.S., typically 4 times.
- Only short periods without dividends.
- Theoretically easily handled if dividends are known.
- But in practice: Not known with certainty.
- For short periods: $S \approx E(D + S_T)$.
- For given S, a higher D means lower S_T, lower c, higher p.
- Intuitive: High D means less left in corporation, thus option to *buy* share at K is less valuable.
- Intuitive: High D means less left in corporation, thus option to *sell* share at K is more valuable.
- Absence-of-arbitrage proofs rely on short sales.
- Short sale of shares: Must compensate for dividends.
- Short sale starts with borrowing share. Must compensate the lender of the share for the dividends missing. (Cannot just hand back share later, neglecting dividends in meantime.)
- When a-o-arbitrage proof involves shares: Could in some cases assume $D = 0$ with full certainty.
- If not $D = 0$ with certainty, conclude with inequalities instead of equalities.

More inequality results on option values

Absence-of-arbitrage proofs for American calls:

1. $C \geq 0$: If not, buy option, keep until expiration. Get something positive now, certainly nothing negative later.
2. $C \leq S$: If not, buy share, sell (i.e., write) call, receive $C - S > 0$. Get $K > 0$ if option is exercised, get S if not.
3. $C \geq S - K$: If not, buy option, exercise immediately.
4. When (for sure) no dividends: $C \geq S - Ke^{-rt}$: If not, do the following:

			Expiration	
	Now	Div. date	If $S_T \leq K$	If $S_T > K$
Sell share	S	0	$-S_T$	$-S_T$
Buy call	$-C$	0	0	$S_T - K$
Lend	$-Ke^{-rt}$	0	K	K
	≥ 0	0	≥ 0	0

A riskless arbitrage.

Important implication: *American call option on shares which certainly will not pay dividends before option's expiration, should not be exercised before expiration,* since

$$C \geq S - Ke^{-rt} > S - K.$$

Worth more "alive than dead." When no dividends: *Value of American call equal to value of European.*

Summing up some results

Both American and European call options on shares which for sure pay no dividends:

$$C \geq S - Ke^{-rt} > S - K.$$

American call options on shares which may pay dividends:

$$C \geq S - K.$$

American calls when dividends possible: More

- For each dividend payment: Two dates.
  - One date for announcement, after which D known.
  - One *ex-dividend* date, after which share does not give the right to that dividend payment.
- Our interest is in ex-dividend dates.
- Owners of shares on morning of ex-div. date receive D.
- Assume there is a given number of ex-div. dates.
- Say, 2 ex-div. dates, t_{d1}, t_{d2}, before option's expiration, T.
- Can show: $C > S - K$ except just before t_{d1}, t_{d2}, T.
- Assume contrary, $C \leq S - K$. Then riskless arbitrage:
- Buy call, exercise just before:

	Now	Just before next t_{di} or T
Buy call	$-C$	$S - K$
Sell share	S	$-S$
Lend	$-K$	$Ke^{r\Delta t}$
	≥ 0	$K(e^{r\Delta t} - 1)$

- Riskless arbitrage, except if $\Delta t \approx 0$, just before.

Implication: *When possible ex-dividend dates are known, American call options should never be exercised except perhaps just before one of these, or at expiration.*

Trading strategies with options, Hull ch. 11[2]

- Consider profits as functions of S_T.
- Can obtain different patterns by combining different options.

Example (Bear spread, Hull Fig. 11.5):[3]

[2] 7th ed.: ch. 10
[3] 7th ed.: Fig. 10.5

Trading strategies, contd.[4]

- Strategies in ch. 11 sorted like this:
  - Sect. 11.1: One option, one share.
  - Sect. 11.2: 2 or 3 calls, or 2 or 3 puts, different K values.
  - End of 11.2, pp. 227–229: Different expiration dates.[5]
  - Sect. 11.3: "Combinations", involving both puts and calls.
- Among these types of strategies, those with different expiration dates cannot be described by same method as others.
- The first, second, and fourth type:
  - Use diagram for values at expiration for each security involved.
  - Payoff at expiration is found by adding and subtracting these values.
  - Net profit is found by subtracting initial outlay from payoff.
  - Initial outlay could be negative (if, e.g., short sale of share).
  - Remember: No exact relationship between payoff and initial outlay is used in these diagrams — will depend upon, e.g., time until expiration, volatility, interest rate.
- For the third type: "Profit diagrams for calendar spreads are usually produced so that they show the profit when the short-maturity option expires on the assumption that the long-maturity option is closed out at that time" (Hull, p. 244).[6]

[4] All references to ch. 11 on this page would be to ch. 10 in the 7th ed. Each section 11.n corresponds to section 10.n.
[5] 7th ed.: pp. 244–245
[6] 7th ed.: p. 228 (almost identical)

Developing an exact option pricing formula

- Exact formula based on observables very useful.
- Most used: Black and Scholes formula.
- Fischer Black and Myron Scholes, 1973.
- Their original derivation used difficult math.
- Continuous-time stochastic processes.
- First here: (Pedagogical tool:) Discrete time.
- Assume trade takes place, e.g., once per week.
- Option pricing formula in discrete time model.
- Then let time interval length decrease.
- Limit as interval length goes to zero.
- Option pricing formula in continuous time.

Assumptions for exact option pricing

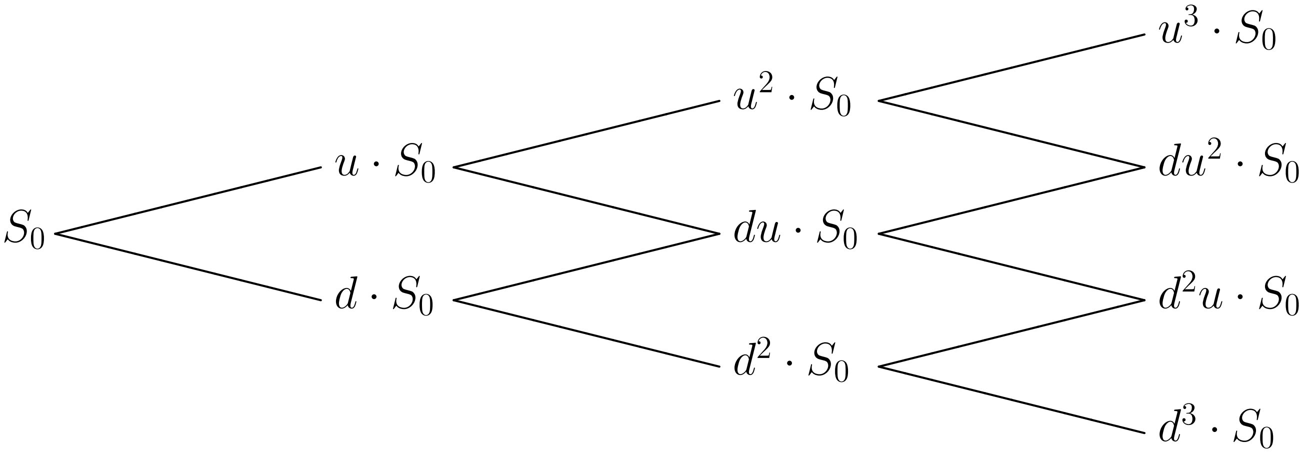
Common assumptions

1. No riskless arbitrage exists.
2. Short sales are allowed.
3. No taxes or transaction costs.
4. Exists a constant risk free interest rate, r.
5. Trade takes place at each available point in time. (Two different interpretations: Once per period, or continuously.)
6. S_{t+s}/S_t is stoch. indep. of S_t and history before t.

Separate assumptions (discrete = d, continuous = c)

7d.	S_{t+1}/S_t has two possible outcomes, u and d. Everyone agrees on these.	7c.	Any sample path $\{S_t\}_{t=0}^T$ is continuous.
8d.	$\Pr(S_{t+1}/S_t = u) = p^*$ for all t.	8c.	$\text{var}[\ln(S_{t+s}/S_t)] = \sigma^2 s$. Everyone agrees on this.
9d.	S_{t+s} has a binomial distribution.	9c.	S_{t+s} has a lognormal distribution.

Discrete time binomial share price process



Define $X_n = S_{t+n}/S_t$. (These are stochastic variables as viewed from time t. Their distributions do not depend on t.)

$$\Pr(X_1 = u) = p^*.$$

For this course we will not go into detail on the following:

$$\Pr(X_n = u^j d^{n-j}) = \frac{n!}{j!(n-j)!} p^{*j}(1-p^*)^{n-j},$$

the binomial probability for exactly j outcomes of one type (here u) with probability p^*, in n independent draws. ($j \leq n$.)

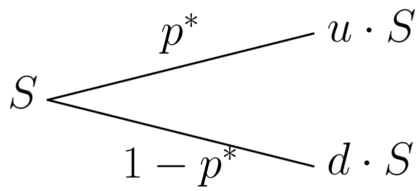
$$\Pr(X_n \geq u^a d^{n-a}) = \sum_{j=a}^{n} \frac{n!}{j!(n-j)!} p^{*j}(1-p^*)^{n-j},$$

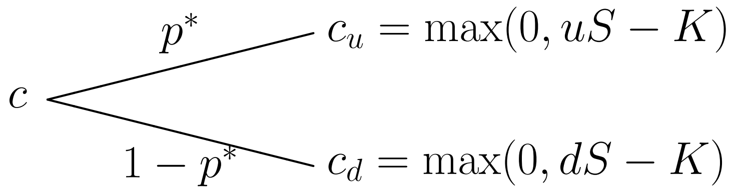
the binomial probability for a or more outcomes of one type (here u) with probability p^*, in n independent draws. ($a \leq n$.)

There is more about this in ch. 20 in Hull,[7] not part of your curriculum.

[7] 7th ed.: ch. 19

Corresponding trees for share and option





- Value of call option with expiration one period ahead?
- "Corresponding trees" mean that option value has upper outcome if and only if share value has upper outcome.
- For any K, know the two possible outcomes for c.
- I.e., for a particular option, c_u, c_d known.
- If $K \leq dS$, then $c_d = dS - K, c_u = uS - K$.
- If $dS < K \leq uS$, then $c_d = 0, c_u = uS - K$.
- If $uS < K$, then $c_d = 0, c_u = 0$.
- This third kind of option is obviously worthless.