ECON4510 – Finance Theory
Lecture 3

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Overview of today's lecture

- Purpose: Construct *Capital Asset Pricing Model,* CAPM
- Equilibrium model for stock market in closed economy
- First: Describe indifference curves in σ_p, μ_p diagram
- Then: Describe opportunity set in the same diagram
  - With only two risky assets
  - With many risky assets
  - With one risky and one risk free asset
  - With many risky and one risk free asset
- Then: Optimal choice based on mean-variance preferences
- Then: Consequences for equilibrium prices (more next time)

Indifference curves in mean-stddev diagrams

(D&D, appendix 6.1)

- If mean and variance are sufficient to determine choices, then mean and $\sqrt{\text{variance}}$ are also sufficient.
- More practical to work with mean (μ) and standard deviation (σ) diagrams.
- Will show that indifference curves are increasing and convex in (σ, μ) diagrams; also slope $\rightarrow 0$ as $\sigma \rightarrow 0^+$.
- Consider normal distribution and quadratic U separately.
- Indifference curves are contour curves of $E[U(\tilde{W})]$.
- Total differentiation:

$$0 = dE[U(\tilde{W})] = \frac{\partial E[U(\tilde{W})]}{\partial \sigma} d\sigma + \frac{\partial E[U(\tilde{W})]}{\partial \mu} d\mu.$$

Indifference curves from quadratic U

Assume $W < -b/(2c)$ with certainty in order to have $U'(W) > 0$.

$$E[U(\tilde{W})] = c\sigma^2 + c\mu^2 + b\mu + a.$$

First-order derivatives are:

$$\frac{\partial E[U(\tilde{W})]}{\partial \sigma} = 2c\sigma < 0, \ \frac{\partial E[U(\tilde{W})]}{\partial \mu} = 2c\mu + b > 0.$$

Thus the slope of the indifference curves,

$$\left.\frac{d\mu}{d\sigma}\right|_{E[U(\tilde{W})] \text{ const.}} = -\frac{\frac{\partial E[U(\tilde{W})]}{\partial \sigma}}{\frac{\partial E[U(\tilde{W})]}{\partial \mu}} = \frac{-2c\sigma}{2c\mu + b},$$

is positive, and approaches 0 as $\sigma \to 0^+$.

Indifference curves from quadratic U, contd.

Second-order derivatives are:

$$\frac{\partial^2 E[U(\tilde{W})]}{\partial \sigma^2} = 2c < 0, \ \frac{\partial^2 E[U(\tilde{W})]}{\partial \mu^2} = 2c < 0, \ \frac{\partial^2 E[U(\tilde{W})]}{\partial \mu \partial \sigma} = 0.$$

The function is concave, thus it is also quasi-concave. (*MA2* sect. 4.7; *FMEA* sect. 2.5.)

Indifference curves from normally distributed $\tilde{W}$

Let $f(\varepsilon) \equiv (1/\sqrt{2\pi})e^{-\varepsilon^2/2}$, the std. normal density function. Let $W = \mu + \sigma\varepsilon$, so that $\tilde{W}$ is $N(\mu, \sigma^2)$.

Define expected utility as a function:

$$E[U(\tilde{W})] = V(\mu, \sigma) = \int_{-\infty}^{\infty} U(\mu + \sigma\varepsilon)f(\varepsilon)d\varepsilon.$$

Slope of indifference curves:

$$-\frac{\frac{\partial V}{\partial \sigma}}{\frac{\partial V}{\partial \mu}} = \frac{-\int_{-\infty}^{\infty} U'(\mu + \sigma\varepsilon)\varepsilon f(\varepsilon)d\varepsilon}{\int_{-\infty}^{\infty} U'(\mu + \sigma\varepsilon)f(\varepsilon)d\varepsilon}.$$

Denominator always positive. Will show that integral in numerator is negative, so minus sign makes the whole fraction positive.

Indifference curves from normal distribution, contd.

Integration by parts:

Observe $f'(\varepsilon) = -\varepsilon f(\varepsilon)$. Thus:

$$\int U'(\mu + \sigma\varepsilon)\varepsilon f(\varepsilon) d\varepsilon = -U'(\mu + \sigma\varepsilon) f(\varepsilon) + \int U''(\mu + \sigma\varepsilon)\sigma f(\varepsilon) d\varepsilon.$$

First term on RHS vanishes in limit when $\varepsilon \to \pm\infty$, so that:

$$\int_{-\infty}^{\infty} U'(\mu + \sigma\varepsilon)\varepsilon f(\varepsilon) d\varepsilon = \int_{-\infty}^{\infty} U''(\mu + \sigma\varepsilon)\sigma f(\varepsilon) d\varepsilon < 0.$$

Another important observation:

$$\lim_{\sigma \to 0^+} \frac{d\mu}{d\sigma} = \frac{-U'(\mu) \int_{-\infty}^{\infty} \varepsilon f(\varepsilon) d\varepsilon}{U'(\mu) \int_{-\infty}^{\infty} f(\varepsilon) d\varepsilon} = 0.$$

Indifference curves from normal distribution, contd.

To show concavity of $V()$:

$$\lambda V(\mu_1, \sigma_1) + (1 - \lambda) V(\mu_2, \sigma_2)$$

$$= \int_{-\infty}^{\infty} [\lambda U(\mu_1 + \sigma_1 \varepsilon) + (1 - \lambda) U(\mu_2 + \sigma_2 \varepsilon)] f(\varepsilon) d\varepsilon$$

$$< \int_{-\infty}^{\infty} U(\lambda \mu_1 + \lambda \sigma_1 \varepsilon + (1 - \lambda) \mu_2 + (1 - \lambda) \sigma_2 \varepsilon) f(\varepsilon) d\varepsilon$$

$$= V(\lambda \mu_1 + (1 - \lambda) \mu_2, \lambda \sigma_1 + (1 - \lambda) \sigma_2).$$

The function is concave, thus it is also quasi-concave.

Mean-variance portfolio choice

- One individual, mean-var preferences, a given W_0 to invest at $t = 0$
- Regards probability distribution of future ($t = 1$) values of securities as exogenous; values at $t = 1$ include payouts like dividends, interest
- Today also: Regards security prices at $t = 0$ as exogenous
- Later: Include this individual in equilibrium model of competitive security market at $t = 0$

Notation: Investment of W_0 in n securities:

$$W_0 = \sum_{j=1}^{n} p_{j0} X_j = \sum_{j=1}^{n} W_{j0}.$$

Mean-variance portfolio choice, contd.

Value of this one period later:

$$\tilde{W} = \sum_{j=1}^{n} \tilde{p}_{j1} X_j = \sum_{j=1}^{n} \tilde{W}_j = \sum_{j=1}^{n} p_{j0} \frac{\tilde{p}_{j1}}{p_{j0}} X_j$$

$$= \sum_{j=1}^{n} p_{j0}(1 + \tilde{r}_j) X_j = \sum_{j=1}^{n} W_{j0}(1 + \tilde{r}_j)$$

$$= W_0 \sum_{j=1}^{n} \frac{W_{j0}}{W_0}(1 + \tilde{r}_j) = W_0 \sum_{j=1}^{n} w_j(1 + \tilde{r}_j) = W_0(1 + \tilde{r}_p),$$

where the w_j's, known as portfolio weights, add up to unity. $\tilde{r}_j = \frac{\tilde{P}_{j1}}{P_{j0}} - 1$ is *rate of return* on asset j

Mean-var preferences for rates of return

$$\tilde{W} = W_0 \sum_{j=1}^{n} w_j(1 + \tilde{r}_j) = W_0 \left(1 + \sum_{j=1}^{n} w_j \tilde{r}_j\right) = W_0(1 + \tilde{r}_p).$$

- $\tilde{r}_p$ is rate of return for investor's portfolio
- If each investor's W_0 fixed, then preferences well defined over $\tilde{r}_p$, may forget about W_0 for now
- Let $\mu_p \equiv E(\tilde{r}_p)$ and $\sigma_p^2 \equiv \text{var}(\tilde{r}_p)$; then:

$$E(\tilde{W}) = W_0(1 + E(\tilde{r}_p)) = W_0(1 + \mu_p),$$

$$\text{var}\,\tilde{W} = W_0^2\,\text{var}(\tilde{r}_p),$$

$$\sqrt{\text{var}(\tilde{W})} = W_0\sqrt{\text{var}(\tilde{r}_p)} = W_0\sigma_p.$$

Mean-var preferences for rates of return, contd.

Increasing, convex indifference curves in $(\sqrt{\text{var}(\tilde{W})}, E(\tilde{W}))$ diagram imply increasing, convex indifference curves in (σ_p, μ_p) diagram

But: A change in W_0 will in general change the shape of the latter kind of curves ("wealth effect")

Mean-var opportunity set, two risky assets

Start with simple case: Investor may construct (any) portfolio of two risky assets. What is opportunity set in (σ_p, μ_p) diagram?

$$W_0 = W_{10} + W_{20} \quad \text{at} \quad t = 0,$$

$$\tilde{W} = W_{10}(1 + \tilde{r}_1) + W_{20}(1 + \tilde{r}_2) = W_0 \left[\frac{W_{10}}{W_0}(1 + \tilde{r}_1) + \frac{W_{20}}{W_0}(1 + \tilde{r}_2) \right]$$

$$= W_0[a(1 + \tilde{r}_1) + (1 - a)(1 + \tilde{r}_2)] \equiv W_0(1 + \tilde{r}_p) \quad \text{at} \quad t = 1.$$

For $j = 1, 2$, let $\mu_j \equiv E(\tilde{r}_j)$, $\sigma_j^2 \equiv \text{var}(\tilde{r}_j)$, and let $\sigma_{12} \equiv \text{cov}(\tilde{r}_1, \tilde{r}_2)$. Then:

$$\mu_p = a\mu_1 + (1 - a)\mu_2 \quad \left(\Rightarrow a = \frac{\mu_p - \mu_2}{\mu_1 - \mu_2} \right),$$

$$\sigma_p^2 = a^2\sigma_1^2 + (1 - a)^2\sigma_2^2 + 2a(1 - a)\sigma_{12}.$$

Mean-var opportunity set, two risky assets, contd.

Taken together, the information on the preceding page gives us σ_p as a function of μ_p:

$$\sigma_p = \sqrt{A\mu_p^2 + B\mu_p + C},$$

where the original five parameters of the problem $(\mu_1, \mu_2, \sigma_1, \sigma_2, \sigma_{12})$ are combined into three new parameters (A, B, C):

$$A \equiv \frac{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}{(\mu_1 - \mu_2)^2},$$

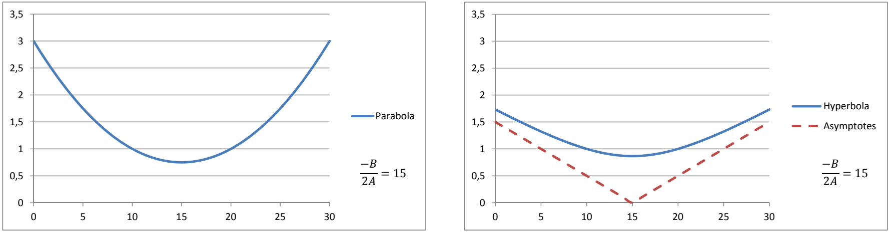
$$B \equiv \frac{-2\mu_2\sigma_1^2 - 2\mu_1\sigma_2^2 + 2\sigma_{12}(\mu_1 + \mu_2)}{(\mu_1 - \mu_2)^2},$$

$$C \equiv \frac{\mu_2^2\sigma_1^2 + \mu_1^2\sigma_2^2 - 2\mu_1\mu_2\sigma_{12}}{(\mu_1 - \mu_2)^2}.$$

In this formulation, the choice parameter, a, has been eliminated.

Mean-var opportunity set, two risky assets, contd.

The function $\sigma(\mu) = \sqrt{A\mu^2 + B\mu + C}$ is called an *hyperbola*, the square root of a *parabola*. Both have minimum points at $\mu = \frac{-B}{2A}$.



D&D, as well as almost all other financial economists, use a transposed version of the diagram to the right, with σ on the vertical axis, μ on the horizontal axis.

Opportunity set, two risky assets, asymptotes

Asymptotes for $\sigma(\mu) = \sqrt{A\mu^2 + B\mu + C}$: Can show that:

$$\mu \to \infty \Rightarrow \sigma \to \sqrt{A}\mu + \frac{B}{2\sqrt{A}}; \;\; \mu \to -\infty \Rightarrow \sigma \to -\sqrt{A}\mu - \frac{B}{2\sqrt{A}}.$$

Proof (of first part only):

$$\lim_{\mu\to\infty}[\sigma(\mu) - \sqrt{A}\mu] = \lim_{\mu\to\infty} \frac{(\sigma(\mu) - \sqrt{A}\mu)(\sigma(\mu) + \sqrt{A}\mu)}{\sigma(\mu) + \sqrt{A}\mu}$$

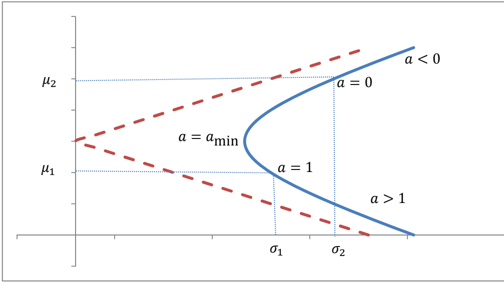
$$= \lim_{\mu\to\infty} \frac{(\sigma(\mu))^2 - A\mu^2}{\sigma(\mu) + \sqrt{A}\mu} = \lim_{\mu\to\infty} \frac{B\mu + C}{\sqrt{A\mu^2 + B\mu + C} + \sqrt{A}\mu}$$

$$= \lim_{\mu\to\infty} \frac{B + \frac{C}{\mu}}{\sqrt{A + \frac{B}{\mu} + \frac{C}{\mu^2}} + \sqrt{A}} = \frac{B}{2\sqrt{A}},$$

and the result follows.

Opportunity set, traced by varying a

- When a varies, the hyperbola is traced out.
- $a = 1$ gives the point (σ_1, μ_1).
- $a = 0$ gives the point (σ_2, μ_2).
- Value of a at minimum point, f.o.c.:



$$0 = \frac{d\sigma^2}{da} = 2a\sigma_1^2 - 2(1-a)\sigma_2^2 + (2-4a)\sigma_{12} \Rightarrow a = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} \equiv a_{\min}.$$

Observe that $a_{\min}$ denotes that value of a which minimizes σ, not the minimum value of a (which is either $-\infty$ or 0, depending on whether short sales are allowed).

Minimum point of hyperbola

Choose notation $\sigma_1 \leq \sigma_2$ (so a is share of portfolio in the less risky asset). Is always $a_{\min} \in [0,1]$? No, will prove $a_{\min} \in [0,\infty)$.

Proof: Define the *correlation coefficient* $\rho_{12} \equiv \frac{\sigma_{12}}{\sigma_1\sigma_2}$. Then:

$$a_{\min} > 1 \iff \sigma_2^2 - \sigma_{12} > \sigma_1^2 + \sigma_2^2 - 2\sigma_{12}$$

$$\iff \sigma_{12} > \sigma_1^2 \iff \rho_{12} > \sigma_1/\sigma_2,$$

which may or may not be true. (Only general restriction on ρ_{12}, known from statistics theory, is $-1 \leq \rho_{12} \leq 1$.) Similarly:

$$a_{\min} < 0 \iff \rho_{12} > \frac{\sigma_2}{\sigma_1} > 1,$$

which is impossible. (In fact, may show $a_{\min} > 0.5$. Will leave this to you as exercise.)

Hyperbola's dependence on correlation

(D&D, Appendix 6.2)

- Five constants determine shape of hyperbola:

$$\mu_1, \mu_2, \sigma_1, \sigma_2, \sigma_{12}.$$

- Assets' coordinates, (σ_1, μ_1) and (σ_2, μ_2), are not sufficient.
- For fixed values of these four, let σ_{12} vary.
- This is *not* something the investor may choose to do, only a way to illustrate different possible shapes of the opportunity set.
- Easier to discuss in terms of $\rho_{12} \equiv \sigma_{12}/\sigma_1\sigma_2$.
- Consider first what hyperbola looks like for $a \in [0, 1]$.
- Consider first the extremes, $\rho_{12} = \pm 1$.
- For $\rho_{12} = 1$ and $a \in [0, 1]$, find $\sigma_p = a\sigma_1 + (1 - a)\sigma_2$.
- Linear in a, thus also in μ_p.
- In interval between the two points: Straight line.
- Line reaches vertical axis somewhere outside interval. Kink.

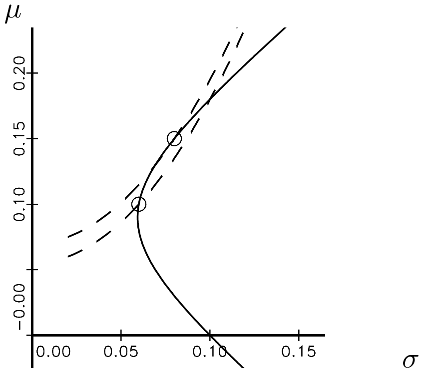
Hyperbola's dependence on correlation

- Opposite extreme, $\rho_{12} = -1$, gives $\sigma_p = \pm[a\sigma_1 - (1-a)\sigma_2]$.
- Also broken line, but now, kink for some $a \in [0, 1]$.
- Specifically, at $a_{\min} = \frac{\sigma_2}{\sigma_1+\sigma_2}$.
- Summing up:
  - For $\rho \in (-1, 1)$, a true (strictly convex) hyperbola.
  - For extreme cases, a broken line.
  - Only for those extreme cases is $\sigma_p = 0$ possible.

  (Illustrated in spreadsheet diagram, not showing asymptotes.)
- Opportunity set consists of the hyperbola or broken line, only.
- When only two risky assets, impossible to obtain point outside (or "inside") hyperbola.

Mean-var portfolio choice, two risky assets

- Increasing, convex indifference curves.
- Increasing, concave opportunity set (upper half).
- Tangency point will maximize (expected) utility.
- Everyone will choose from upper half of hyperbola.
- Called *efficient set.*
- Choice within efficient set depends on preferences.
- More risk averse: Lower σ.



Mean-var opportunity set: n risky assets, $n > 2$

Let variance-covariance matrix of $(\tilde{r}_1, \ldots, \tilde{r}_n)$ be

$$V = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}.$$

Symmetric, $\sigma_{n1} = \text{cov}(\tilde{r}_n, \tilde{r}_1) = \text{cov}(\tilde{r}_1, \tilde{r}_n) = \sigma_{1n}$.

Diagonal elements were previously called $\sigma_i^2 = \text{var}(\tilde{r}_i)$.

If V has full rank, then impossible to construct portfolio of these assets with $\sigma_p = 0$. (No proof now.) Will concentrate on this case. (If less than full rank, $\sigma_p = 0$ can be obtained. Cf., for $n = 2$, the cases $\rho_{12} = \pm 1$.)

Assume now V has full rank. Can be shown: Opportunity set now consists of an hyperbola and the points inside it. (Blackboard.)

Informally discussed in D&D p. 155, formally on pp. 217–222.[1]

[1] 2nd ed.: p. 101, pp. 127–132.

Mean-var opportunity set: n risky assets

For any μ_p, the agents will want as low σ_p as possible:

$$\min_{w_1,\ldots,w_n} \sigma_p \text{ given } \mu_p.$$

This defines the hyperbola called the *frontier portfolio set.* For any σ_p, the agents will want as high μ_p as possible:

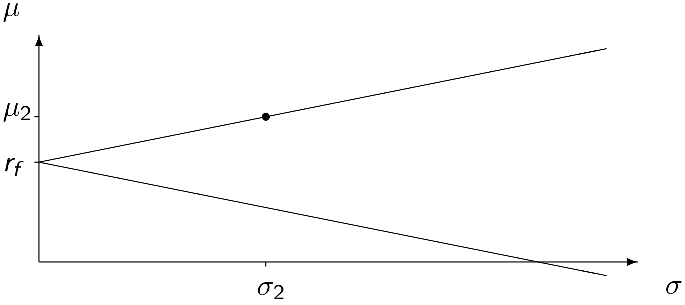
$$\max_{w_1,\ldots,w_n} \mu_p \text{ given } \sigma_p.$$

This defines the upper half of the hyperbola. The upper half of the *frontier portfolio set* is known as the *efficient portfolio set.*

Again: *Efficient* means that part of the opportunity set from which the agents will choose, irrespective of their preferences, but within which we cannot predict their choice, since we do not specify their preferences in any more detail.

Mean-var opport. set: One risky, one risk free asset

- Let $\sigma_1 = \sigma_{12} = 0$ in formulae.
- Get linear relation between σ_p and a; also between σ_p and μ_p.
- Simplifies. Good reason for working with (σ, μ), not (σ^2, μ).
- Opportunity set broken line. Again, upper half is efficient.



Mean-var oppo. set: One risk free, n risky assets, $n > 2$

- Let r_f be rate of return on risk free asset.
- Risk free asset can be combined with any portfolio of risky.
- Everyone will want max μ_p for any given σ_p.
- Assume r_f less than μ at min point.
- (Will return to opposite possibility later.)
- Then: Combination of risk free asset with *tangency portfolio* is efficient. Efficient set is *linear.*
- (Blackboard.)

Mean-var portfolio choice: 1 risk free, n risky assets

- Consider now situation with many agents.
- Assume all believe in same means, variances, covariances.
- With mean-variance preferences, all want some combination of risk free asset with *same* portfolio of risky assets, tangency.
- Straight line efficient set.
- Preferences determine preferred location along line.
- Higher risk aversion: Closer to risk free asset.
- Lower risk aversion: Above tangency portfolio: Borrow money (short sell risk free asset) and invest more than W_0 in tangency portfolio.
- "Two-fund spanning": Restriction of opportunity set to r_f and tangency portfolio is just as good as original opportunity set.
- "Separation" of portfolio composition: May leave to a fund manager to make tangency portfolio available.

Equilibrium condition

- Everyone demands same combination of risky assets.
- Necessary condition for equilibrium: This is equal to supply.
- Agent h splits $W_0^h = W_{0f}^h + W_{0M}^h$.
- W_{0f}^h in risk free asset, possibly negative.
- W_{0M}^h in tangency portfolio, strictly positive. (Why?)
- Tangency portfolio has weights $w_{1M}, \ldots, w_{nM}$.
- Per definition $\sum w_{jM} = 1$.

Equilibrium condition, contd.

- Total demand for n risky assets written as vector:

$$\sum_{h=1}^{H} \begin{bmatrix} w_{1M} W_{0M}^h \\ \vdots \\ w_{nM} W_{0M}^h \end{bmatrix} = \begin{bmatrix} w_{1M} \\ \vdots \\ w_{nM} \end{bmatrix} \sum_{h=1}^{H} W_{0M}^h.$$

- Total has same value composition as each part.
- This must also be value composition of supply.
- Observable, "market portfolio."
- "Portfolio" here means a vector of weights, summing to one.
- The word "portfolio" may sometimes mean some money amount invested in each of the n assets, a vector not summing to one.

CML, market price of risk

- Everyone combines risk free asset and market portfolio.
- Line through $(0, r_f)$ and (σ_M, μ_M) called *Capital Market Line,* CML,

$$\mu_P = r_f + \frac{\mu_M - r_f}{\sigma_M}\sigma_p.$$

  (Blackboard.)
- Slope, $\frac{\mu_M - r_f}{\sigma_M}$, sometimes called *market price of risk.*
- Shows how much must be given up in expected portfolio rate of return in order to reduce standard deviation by one unit.
- All agents have MRS between μ_p and σ_p equal to this.
- Will soon see: This is relevant concept for comparing whole portfolios, but not for individual assets.

Motivating CAPM: Covariances important

- Next derive most important formula in this part of course.
- Model known as the *Capital Asset Pricing Model.* This name also used for the main formula. Formula also called the *Security Market Line* (SML).
- Shows what determines prices of individual assets.
- First motivation: Covariances important.
- Comparing alternative portfolios, when only one of them can be chosen, have assumed *variances* of rates of return are the relevant measure of risk.
- But for individual assets, which can be combined in portfolios, the relevant measure turns out to be a *covariance* with other rates of return.

Covariances important, contd.

- Make two simple, motivating arguments first, without reference to any equilibrium model.
- Consider making an *equally weighted* portfolio of n assets, i.e., with all $w_j = 1/n$. Assume that among the rates of return, one has the maximum variance, $\sigma^2_{\max}$. Then:

$$\lim_{n\to\infty} \sigma_p^2 = \bar{\sigma}_{ij},$$

the average covariance between rates of return, and

$$\lim_{n\to\infty} \frac{\partial \sigma_p^2}{\partial w_i} = 2\bar{\sigma}_{ij}.$$

Proof of motivating results

Observe that:

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}.$$

An equally weighted portfolio has

$$\sigma_p^2 = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij} = \frac{1}{n^2}\sum_{i=i}^{n}\sigma_i^2 + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j\neq i}\sigma_{ij}.$$

Observe that the first term satisfies

$$\frac{1}{n^2}\sum_{i=i}^{n}\sigma_i^2 < \frac{1}{n^2}\cdot n\cdot\sigma_{\max}^2 \to 0 \quad \Leftarrow \quad n\to\infty.$$

The second term satisfies

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j\neq i}\sigma_{ij} = \frac{n^2-n}{n^2}\bar{\sigma}_{ij} \to \bar{\sigma}_{ij} \quad \Leftarrow \quad n\to\infty,$$

which proves the first result.

Proof of motivating results, contd.

Observe next that for any portfolio,

$$\frac{\partial \sigma_p^2}{\partial w_i} = 2w_i\sigma_i^2 + 2\sum_{j\neq i} w_j\sigma_{ij}.$$

Evaluated where all $w_i = 1/n$, this becomes

$$2\frac{\sigma_i^2}{n} + 2\frac{n-1}{n}\bar{\sigma}_{ij} \rightarrow 2\bar{\sigma}_{ij} \quad \Leftarrow \quad n \rightarrow \infty.$$

Derivation of CAPM formula

- Consider an equilibrium, everyone holds combination of risk free asset and market portfolio.
- Will derive relation between μ_j, σ_j (of any asset, numbered j) and the economy-wide variables r_f, μ_M, σ_M.
- As a *thought experiment* (only), make a portfolio with a fraction a in asset j and a fraction $1 - a$ in the market portfolio.
- (Possible, even though M already contains j.)
- For this portfolio p we have:

$$\mu_p = a\mu_j + (1-a)\mu_M, \quad \frac{\partial \mu_p}{\partial a} = \mu_j - \mu_M,$$

$$\sigma_p = \sqrt{a^2\sigma_j^2 + (1-a)^2\sigma_M^2 + 2a(1-a)\sigma_{jM}},$$

$$\frac{\partial \sigma_p}{\partial a} = \frac{a\sigma_j^2 - (1-a)\sigma_M^2 + (1-2a)\sigma_{jM}}{\sqrt{a^2\sigma_j^2 + (1-a)^2\sigma_M^2 + 2a(1-a)\sigma_{jM}}},$$

$$\left.\frac{\partial \sigma_p}{\partial a}\right|_{a=0} = \frac{\sigma_{jM} - \sigma_M^2}{\sigma_M}.$$

Illustrating the derivation

(D&D, sect. 8.2, app. 8.1, fig. 8.1;[2] blackboard)

- Small hyperbola goes through M (i.e., (σ_M, μ_M)).
- At M it has same tangent as large hyperbola: If not, it would have to cross over large hyperbola. But that cannot happen, since large hyperbola is frontier, and j was already available when large hyperbola was formed as frontier.
- The tangent *is* the capital market line.
- Next: Use the equality of these slopes.

[2] 2nd ed.: sect. 7.2, app. 7.2, fig. 7.1.

Derivation of CAPM formula

Use the formula:

$$\frac{d\mu}{d\sigma}\frac{\partial\sigma}{\partial a} = \frac{\partial\mu}{\partial a} \iff \frac{d\mu}{d\sigma} = \frac{\frac{\partial\mu}{\partial a}}{\frac{\partial\sigma}{\partial a}}.$$

Use partial derivatives just found, evaluate at $a = 0$:

$$\left.\frac{\partial\sigma}{\partial a}\right|_{a=0} = \frac{\sigma_{jM} - \sigma_M^2}{\sigma_M}.$$

Plug in and find:

$$\left.\frac{d\mu}{d\sigma}\right|_{a=0} = \frac{\mu_j - \mu_M}{(\sigma_{jM} - \sigma_M^2)/\sigma_M}.$$

This slope of small hyperbola must equal slope of CML:

$$\frac{\mu_j - \mu_M}{(\sigma_{jM} - \sigma_M^2)/\sigma_M} = \frac{\mu_M - r_f}{\sigma_M} \iff \mu_j = r_f + (\mu_M - r_f)\frac{\sigma_{jM}}{\sigma_M^2}.$$

Known as the CAPM equation or the Security Market Line.

Derivation of CAPM formula

Define $\beta_j \equiv \frac{\sigma_{jM}}{\sigma_M^2}$. Then rewrite as

$$E(\tilde{r}_j) - r_f = \beta_j(E(\tilde{r}_M) - r_f),$$

"the expected *excess* rate of return on asset j equals its beta times the expected excess rate of return on the market portfolio."