ECON4510 – Finance Theory
Lecture 11

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Stochastic processes

- These are stochastic variables which evolve over time.
- Some of you may know about these from
  - time series econometrics,
  - other applications in microeconomics or macroeconomics.
- Purpose here: Analyze prices of stocks and options.
- Binomial tree example of stochastic process in discrete time.
- "Discrete time:" Process only defined at certain time points.
- Black-Scholes-Merton option values based on another process.

Stochastic processes, contd.

- In continuous time, i.e., stock values S_t change continuously.
- (Although we typically observe only at some points in time.)
- Also continuous-valued, i.e., S_t can be any positive number.
- (In typical markets, S_t only has two or three decimals.)
- Could just define that process directly.
- Will instead follow Hull, 9th ed., ch. 14.[1]
- First some rather simple, motivating points.
- Will then develop motivation for more complications.

[1] 8th ed., ch. 13, 7th ed., ch. 12

The Markov property

- S_t called a *Markov* process if (the *Markov* property:) the probability distribution of all $S_{t+\Delta t}$ for all later dates $t + \Delta t$, as seen from date t, depends on S_t only.
- For instance, if S_t is a given number, knowledge of particularly high outcomes for S_{t-2} and S_{t-1},or for $S_{t-0.2}$ and $S_{t-0.1}$, will not affect the probability distribution of $S_{t+0.1}$ or $S_{t+0.2}$ or ....
- Alternatively, we could think that the probability distribution of $S_{t+\Delta t}$ could depend on the whole history of S's, or some part of it, say S_{t-s} for some interval before t. Not Markov.

The Markov property, contd.

- One possible type of dependence, called momentum, is that a falling sequence $S_{t-2} > S_{t-1} > S_t$ increases the probability of an outcome S_{t+1} less than S_t (i.e., most likely, the fall continues). This is not Markov. For a Markov process, a rising sequence $S_{t-2} < S_{t-1} < S_t$ will, if it has the same value for S_t, imply exactly the same probability distribution for S_{t+1} as the falling sequence $S_{t-2} > S_{t-1} > S_t$.
- Exist many types of Markov processes, with many different types of probability distributions for, e.g., S_{t+1} conditional on S_t.
- "Markov processes" should thus be viewed as a wide class of stochastic processes, with one particular common characteristic, the Markov property.

The Markov property, economic implications

- Connection to weak-form market efficiency.
- All available information reflected in today's S_t.
- Probabilities of future $S_{t+\Delta t}$ depend on S_t.
- But historical S values cannot matter.
- Implication of $S_{t-\Delta t}$ for $S_{t+\Delta t}$? Already in S_t.

Implications of Markov property for variance

- Markov: $S_2 - S_1$ is stochastically independent of $S_1 - S_0$.
- Also $S_3 - S_2$, etc.
- Assume we are at time 0, know S_0.
- Can write $S_2 = S_0 + (S_1 - S_0) + (S_2 - S_1)$.
- As seen from time 0, S_0 has no variance.
- Then:

$$\text{var}(S_2) = \text{var}[(S_2 - S_1) + (S_1 - S_0)] = \text{var}(S_2 - S_1) + \text{var}(S_1 - S_0).$$

- The last equality is due to stochastic independence.

Implications for variance, contd.

- Assume all changes $S_{t+1} - S_t$ have same variance.
- Then $\text{var}(S_2) = \text{var}(S_2 - S_1) + \text{var}(S_1 - S_0) = 2\,\text{var}(S_{t+1} - S_t)$.
- More precisely, introduce conditional variance, given S_0.
- $\text{var}(S_2|S_0) = 2\,\text{var}(S_{t+1} - S_t)$.
- Likewise: $\text{var}(S_3|S_0) = 3\,\text{var}(S_{t+1} - S_t)$.
- Generally: $\text{var}(S_T|S_0) = T\,\text{var}(S_{t+1} - S_t)$.
- Implication: (conditional) variance proportional to time.
- Standard deviation proportional to square root of time.
- (In what follows, like Hull, use $\tilde{X} \sim \phi(E(\tilde{X}), \text{var}(\tilde{X}))$ to indicate that $\tilde{X}$ has a normal distribution, but use $N(x) = \Pr(\frac{\tilde{X} - E(\tilde{X})}{\sqrt{\text{var}(\tilde{X})}} \leq x)$ to denote the standard normal cumulative distribution function.)

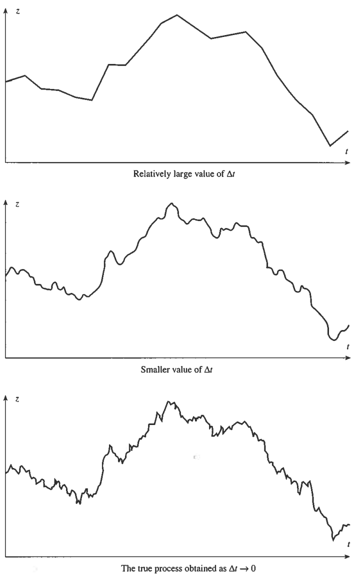
Wiener processes (also called Brownian motion)

- So far, in addition to the Markov property, have assumed the variance of changes is the same for different periods.
- Assume now in addition that $\text{var}(S_{t+1} - S_t | S_t)$ equals 1, and that the expected change $E(S_{t+1} - S_t | S_t)$ equals 0.
- (A bit like looking at a standardized distribution, like $\phi(0, 1)$. Will call this process z_t (or sometimes $z(t)$), not S_t.)
- This gives us a particular type of Markov process called a *Wiener* process, defined by two properties. z_t is a Wiener process if and only if both are satisfied:
  - The change Δz during a short time interval Δt is $\Delta z = \epsilon\sqrt{\Delta t}$, where ϵ has a standard normal (Gaussian) distribution (with $E(\epsilon) = 0, \ \text{var}(\epsilon) = 1)$.
  - The values of Δz for non-overlapping intervals Δt are stochastically independent.

Wiener processes, contd.

- Over longer interval, $z(T) - z(0)$ is normally distributed, the sum of N changes over intervals of length Δt, i.e., $N\Delta t = T$; $z(T) - z(0) = \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t}$.
- This implies $E(z(T) - z(0)) = 0$, $\text{var}(z(T) - z(0)) = N\Delta t = T$. These do not depend on the length of Δt.
- In limit when $\Delta t \to 0$, dz is change during dt; $\text{var}(dz) = dt$.
- See Fig. 14.1 in Hull 9th (13.1 in 8th, 12.1 in 7th).

Figure 13.1 How a Wiener process is obtained when $\Delta t \to 0$ in equation



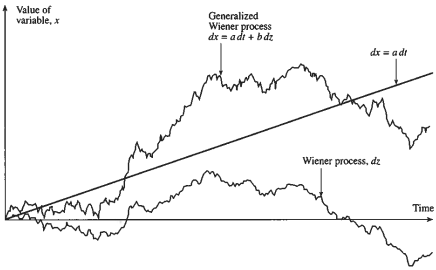
Generalized Wiener processes

- First multiply the Wiener process dz by a constant, b.
- $b\,dz$ has variance $b^2\,\text{var}(dz) = b^2 dt$.
- Then allow for an expected change different from zero,

$$dx = a\,dt + b\,dz$$

- This amounts to adding a non-stochastic linear growth path to the stochastic $b\,dz$, and is illustrated in Fig. 14.2 in Hull 9th (13.2 in 8th, 12.2 in 7th).

Figure 13.2 Generalized Wiener process with $a = 0.3$ and $b = 1.5$.



Generalized Wiener processes, contd.

- The *generalized Wiener process* X is normally distributed with

$$E(X(T) - X(0)|X(0)) = aT,$$

$$\text{var}(X(T) - X(0)|X(0)) = b^2 T.$$

- The process is also called *Brownian motion with drift.*

Generalized Wiener processes; Itô processes

- A further generalization: Allow a and b to depend on (x, t),

$$dx = a(x, t)dt + b(x, t)dz.$$

- This is called an *Itô process.* In general not normally distributed.
- Over a small time interval Δt we get

$$\Delta x \approx a(x, t)\Delta t + b(x, t)\epsilon\sqrt{\Delta t}.$$

- For non-overlapping intervals the changes in x are stochastically independent, so all Itô processes are Markov processes.

Stochastic process for a stock price

- Looking for something more realistic than the binomial tree.
- Expected change will not be zero, so cannot use Wiener process.
- Could we use generalized Wiener process?
- Expected change over interval of length T is aT.
- Suppose $S_0 = 10$, $a = 1$, and that T is measured in years.
- Expected stock price in ten years is $E(S_{10}|S_0 = 10) = 20$.
- Expected stock price ten years later, $E(S_{20}|S_0 = 10) = 30$.
- Also, if S_{10} equals its expectation, $E(S_{20}|S_{10} = 20) = 30$.

Stochastic process for a stock price, contd.

- But the expected growth rate over the time interval $(10, 20)$ is substantially lower than the expected growth rate over $(0, 10)$, since growth rates are relative numbers, and $30/20 < 20/10$.
- More likely shareholders require constant expected growth rate.
- Need exponential expected path, not linear expected path.
- Will obtain this by letting $E(dS) = \mu\, S\, dt$.
- For the non-stochastic part (or, if $\sigma = 0$): $\frac{dS}{dt} = \mu\, S$.
- Integrating between 0 and T: $S_T = S_0 e^{\mu T}$ when $\sigma = 0$.
- This leads to a suggestion of

$$dS = \mu\, S\, dt + \sigma\, dz$$

or, better,

$$dS = \mu\, S\, dt + \sigma\, S\, dz.$$

Stochastic process for a stock price, contd.

- From previous slide: a suggestion of

$$dS = \mu\, S\, dt + \sigma\, dz$$

or

$$dS = \mu\, S\, dt + \sigma\, S\, dz.$$

- Choose the latter so that a relative change in S not only has a constant expected value, $\mu\, dt$, but also a constant variance, $\sigma^2 dt$,

$$\frac{dS}{S} = \mu\, dt + \sigma\, dz.$$

- This stock price process process is basis for the most widespread option pricing theories, like the one in ch. 15 of Hull (9th ed.), Black-Scholes-Merton (8th ed., ch. 14, 7th ed., ch. 13).
- The process is called *geometric Brownian motion with drift.*

Stochastic process for a stock price, contd.

- Since S appears on right-hand side in dS formula: Not a generalized Wiener process, but a bit more complicated.
- dS is an Itô process, with $a(S, t) = \mu S$ and $b(S, t) = \sigma S$.
- Different stocks will differ in μ and/or σ.
- Stock i has constants μ_i, σ_i, stock j has constants μ_j, σ_j
- Hull discusses these variables in section 14.4 (9th ed.).[2]
- Remember: Hull's book does not rely on the CAPM.
- Imprecise discussion of how μ depends on r_f and risk.
- Footnote[3] 5, p. 311, 9th ed., means μ depends on covariance, not on σ.

[2] 8th ed., sect. 13.4, 7th ed., sect. 12.4

[3] 8th ed., fn. 4, p. 289, 7th ed., fn. 4, p. 268

Functions of Itô processes

- When x is an Itô process, $dx = a(x,t)dt + b(x,t)dz$:
  - Is a function G of x also an Itô process?
  - If yes, what happens to the functions $a(x,t)$ and $b(x,t)$?
  - Put differently: G will also have functions like these.
  - What do the two functions look like for G?
- Motivation: Call option value as function of S.
- Find this via a general rule, Itô's lemma.
- A bit more complicated than suggested above.
- Call option not only function of S; also of t.
- Option's value depends on time until expiration.
- For some given S, different t's give different c's.
- Thus, the more general questions are:
  - If x is an Itô process, is $G(x,t)$ an Itô process?
  - If yes, what do the "a and b functions" look like for G?
- The answers are given by *Itô's lemma.*
- Will not prove this mathematically.
- But will show how and why it differs from usual differentiation.

Itô's lemma

- Assume x is an Itô process:
- $dx = a(x, t)dt + b(x, t)dz$, where z is a Wiener process.
- Then $G(x, t)$ is also an Itô process:

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right) dt + \frac{\partial G}{\partial x}b\, dz.$$

- We recognize the general form of an Itô process.
- The expression above is Hull's equation[4] (14.12) (9th ed.).
- In fact, this is short-hand, dropping arguments.

[4] 8th ed., eq. (13.12), 7th ed., eq. (12.12)

Itô's lemma, contd.

- Contains six different functions of (x, t).
- Both a, b, G, and the partial derivatives of G.
- Right-hand side should really be written like this:

$$\left(\frac{\partial G(x,t)}{\partial x}a(x,t) + \frac{\partial G(x,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 G(x,t)}{\partial x^2}[b(x,t)]^2\right) dt + \frac{\partial G(x,t)}{\partial x}b(x,t)dz.$$

- Perhaps this looks complicated, but:
- In our applications, G, a, and b are fairly simple.

Why not ordinary differentiation? Hull, p. 319f (9th ed.)

(8th ed., p. 297f, 7th ed., p. 275 f)

- Approximation of a function by its tangent:

$$\Delta G \approx \frac{dG}{dx}\Delta x$$

  when G is a function of one variable, x.
- Holds precisely in limit as $\Delta x \to 0$.
- As long as $\Delta x \neq 0$, can use Taylor series expansion:

$$\Delta G = \frac{dG}{dx}\Delta x + \frac{1}{2}\frac{d^2 G}{dx^2}\Delta x^2 + \frac{1}{6}\frac{d^3 G}{dx^3}\Delta x^3 + \dots$$

- As $\Delta x \to 0$, higher-order terms vanish.
- $G(x, y)$, two dimensions, a tangent plane:

$$\Delta G \approx \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y.$$

Why not use ordinary differentiation, contd.

- When both Δx and $\Delta y \neq 0$, can use Taylor series:

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x \partial y}\Delta x\ \Delta y + \frac{1}{2}\frac{\partial^2 G}{\partial y^2}\Delta y^2 + \dots$$

- Again, precisely in limit as $\Delta x \to 0$ and $\Delta y \to 0$:

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy.$$

- Want to find a similar expression for Itô processes.
- But all higher-order terms do not vanish.

Itô's lemma vs. ordinary differentiation

- Assume x is an Itô process:
- $dx = a(x,t)dt + b(x,t)dz$, where z is a Wiener process.
- Let G be a function $G(x,t)$, and use Taylor expansion:

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x \partial t}\Delta x\, \Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \ldots$$

- Only novelty here: Have called second variable t, not y.
- When $\Delta x \to 0$, need to observe the following.
- $\Delta x = a\,\Delta t + b\epsilon\sqrt{\Delta t}$ implies:
- $(\Delta x)^2 = b^2\epsilon^2\,\Delta t$ + terms of higher order.
- Since Δx contains a $\sqrt{\Delta t}$ term, normal rules don't work.
- Must include extra term with second-order partial derivative.
- The extra term contains ϵ^2, and ϵ is stochastic.

Itô's lemma vs. ordinary differentiation, contd.

- Hull explains why $E(\epsilon^2 \Delta t) = \Delta t$.
- Hull also explains that $\text{var}(\epsilon^2 \Delta t)$ is of order $(\Delta t)^2$.
- Variance approaches zero fast as $\Delta t \to 0$.
- Thus: In limit $\epsilon^2 \Delta t$ is nonstochastic, $= \Delta t$.
- This gives us the following formula in the limit:

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 dt$$

- Insert for dx from above to find the form we used above:

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b \, dz.$$

Example of application of Itô's lemma

- Consider the stock price process from slide 16:
- Assume $dS = \mu S\, dt + \sigma S\, dz$; z is a Wiener process.
- What kind of process is $\ln S$?
- Natural question; deterministic part of S is exponential in t.
- Might believe that deterministic part of $\ln S$ is linear in t.
- Observe this application of Itô's lemma is fairly simple:
  - "$a(S, t)$ function" of S process is μS. Simple, and no t.
  - "$b(S, t)$ function" of S process is σS. Simple, and no t.
  - The $G(S, t)$ function is $\ln S$. Fairly simple, and no t.
- Know from Itô's lemma that $\ln S$ is an Itô process.
- But what are the "a and b functions" of the G process?
- Will turn out that they are very simple. Constants, no S, no t.
- But slightly less simple than one might have thought.
- The constant which multiplies dt is not μ.
- Would be natural suggestion based on deterministic $S_T = S_0 e^{\mu T}$.

Example; lognormal property, Hull, sect. 14.7 (9th ed.)

(8th ed., sect. 13.7, 7th ed., sect. 12.6)

- With $G(S,t) \equiv \ln S$, need three partial derivatives:

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \quad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \quad \frac{\partial G}{\partial t} = 0.$$

- Then Itô's lemma says that:

$$dG = \left(\frac{1}{S}\mu S + 0 + \frac{1}{2}\left(-\frac{1}{S^2}\right)(\sigma S)^2\right) dt + \frac{1}{S}\sigma S\ dz$$

$$= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\ dz.$$

- So this is an Itô process with constant a and b functions.
- Implies that $\ln S$ is a generalized Wiener process.
- Can use formulae from slide 12.

Example, contd.

- The change $\ln S_T - \ln S_0$ is normally distributed:

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right],$$

which implies (by adding the known $\ln S_0$)

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right].$$

- $\ln S$ is normally distributed.
- By definition then, S is lognormally distributed.
- Not obvious earlier, but by using Itô's lemma.

The lognormal distribution of stock prices

- On slide 15, required an exponential expected path, $S_T = S_0 e^{\mu T}$.
- Could thus not use the generalized Wiener process for S.
- (Would have implied S having a normal distribution.)
- Found instead something similar for relative changes in S,

$$\frac{dS}{S} = \mu\, dt + \sigma\, dz.$$

- This implies S is lognormal, $\ln(S)$ is normal.
- Relation between these two distributions may be confusing.
- Remember that $\ln(S)$ is not linear, thus $E[\ln(S)] \neq \ln[E(S)]$:
  - $E[\ln(S_T)|S_0] = \ln(S_0) + (\mu - \sigma^2/2)T$,
  - $E(S_T|S_0) = S_0 e^{\mu T}$ so that $\ln[E(S_T|S_0)] = \ln(S_0) + \mu T$.

The lognormal distribution of stock prices

- The variance expression is simpler for $\ln(S_T)$ than for S_T:
  - $\text{var}[\ln(S_T)|S_0] = \sigma^2 T$,
  - $\text{var}(S_T|S_0) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$.
- Footnote 2 on p. 323 in Hull (9th ed.)[5] refers to a note on this:

http://www-2.rotman.utoronto.ca/∼hull/technicalnotes/TechnicalNote2.pdf

- $S_T = S_0 e^{xT}$ defines continuously-compounded rate of return x.
- Its distribution is $x \sim \phi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right)$.

[5] 8th ed., fn. 2, p. 301, 7th ed., fn. 2, p. 279

Monte Carlo simulation (Hull sections 14.3 and 21.6)

(8th ed., sect. 13.3 and 20.6, 7th ed., sect. 12.3 and 19.6)

- From lecture 10: Can find option values as expectations.
  - Not based on actual probabilities, but on "risk neutral" probabilities.
- Next lecture finds $c(S, K, T, r, \sigma)$ from lognormal distribution.
- Numerically, e.g., $c(10, 8, 2, 0.05, 0.2)$: Exists alternative method.
- For complicated nonlinear functions: Use Monte Carlo simulation:
  - Computer draws numbers, S_T, from a probability distribution.
  - Typically thousands of independent drawings from same distribution.
  - Gives *frequency distribution,* similar to probability distribution.
  - For each draw, compute some function of it, e.g., $\max(0, S_T - K)$.
  - Average for, e.g., 10 000 draws gives estimate of $E[\max(0, S_T - K)]$.
  - Could also calculate, e.g., $\text{var}[\max(0, S_T - K)]$, but less interesting.
  - Expectation gives option value if use risk neutral probabilities for S_T.
    - ⋆ Just need to take present value, $E[\max(0, \hat{S}_T - K)]e^{-rT}$.
  - Can also be done for many periods, and for functions of many variables.

Monte Carlo simulation in Excel spreadsheet

- Consider $c(10, 8, 2, 0.05, 0.2)$ when stock price is lognormal.
- First determine parameters of probability distribution of $\ln(\hat{S}_2)$.
- For "risk neutral" process, should let $\mu = r$.
- Use $S_0 = 10$, and observe that $\sigma = 0.2 \Rightarrow \sigma^2 = 0.04$.
- Use these μ, S_0, σ^2 in formula from slide 27,

$$\ln \hat{S}_2 \sim \phi \left[\ln(10) + \left(0.05 - \frac{0.04}{2} \right) \cdot 2, 0.04 \cdot 2 \right].$$

- Create lognormal sample using Excel's RAND, NORMSINV, and EXP.
- For each S_T in sample, calculate function values using Excel's MAX.
- Across sample, estimate expectation using Excel's AVERAGE.
- Next time: Exact formula, may then compare results with M-C.

Monte Carlo simulation in Excel, contd.

Column B contains sample of 100 numbers uniformly distributed on $[0, 1]$.

	A	B	C	D	E	F	G
1		Uniform rand. no.s	Normally distributed	Lognormal price	Call option	Input data	
2	Average of below	=AVERAGE(B3:B102)	=AVERAGE(C3:C102)	=AVERAGE(D3:D102)	=AVERAGE(E3:E102)		
3		=RAND()	=NORMSINV(B3)*G	=EXP(C3)	=MAX(0;D3-$G$4)	S0 =	10
4		=RAND()	=NORMSINV(B4)*G	=EXP(C4)	=MAX(0;D4-$G$4)	K =	8
5		=RAND()	=NORMSINV(B5)*G	=EXP(C5)	=MAX(0;D5-$G$4)	T =	2
6		=RAND()	=NORMSINV(B6)*G	=EXP(C6)	=MAX(0;D6-$G$4)	r =	0,05
7		=RAND()	=NORMSINV(B7)*G	=EXP(C7)	=MAX(0;D7-$G$4)	sig =	0,2
8		=RAND()	=NORMSINV(B8)*G	=EXP(C8)	=MAX(0;D8-$G$4)		
9		=RAND()	=NORMSINV(B9)*G	=EXP(C9)	=MAX(0;D9-$G$4)	Output call	
10		=RAND()	=NORMSINV(B10)*G	=EXP(C10)	=MAX(0;D10-$G$4)	option val. =	
11		=RAND()	=NORMSINV(B11)*G	=EXP(C11)	=MAX(0;D11-$G$4)	=$E$2*EXP(-	
12		=RAND()	=NORMSINV(B12)*G	=EXP(C12)	=MAX(0;D12-$G$4)		

NORMSINV applied to uniform distribution gives normal distribution.

The full contents of cell C3, which gives the normally distributed $\ln(S_2)$:

```
=NORMSINV(B3)*$G$7*SQRT($G$5)+LN($G$3)+$G$5*($G$6-0,5*$G$7^2)
```

The full contents of cell F11, which gives the c_0, the call option value:

```
=$E$2*EXP(-$G$6*$G$5)
```

(This is Norwegian; comma as decimal sign; semicolon as separator.)

Monte Carlo: How obtain the desired distribution?

- Statement from previous page needs explanation:

"NORMSINV applied to uniform distribution gives normal distribution."

- Excel's RAND() gives uniform random numbers; but we need normal.
- Function to use is the inverse of the cumulative distribution function.
- (Works also for other distributions than normal, if know inverse cdf.)
- For standard normal, this inverse is the Excel function NORMSINV.
- Cumulative distribution function for standard normal is N in Hull.
- Defined by $N(x) = \Pr(\tilde{X} \leq x)$ when $\tilde{X} \sim \phi(0,1)$, standard normal.
- N is monotonically increasing, continuous, thus has inverse $N^{-1}(u)$.
- If $\tilde{U}$ is uniform on $[0,1]$, then $\tilde{X} \equiv N^{-1}(\tilde{U})$ is standard normal.
- Proof: $\Pr[N^{-1}(\tilde{U}) \leq x] = \Pr[\tilde{U} \leq N(x)] = N(x)$.
- First equality follows since the monotonic N is applied to both sides.
- Second equality is a property of $\tilde{U}$; its cdf is $F(u) = u$ when $u \in [0,1]$.