ECON4510 – Finance Theory
Lecture 12

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The Black-Scholes-Merton formula (Hull 9th, 15.5–15.8)

(8th ed., 14.5–14.8, 7th ed., 13.5–13.8)

- Assume S_t is a geometric Brownian motion with drift.
  - Let $\sigma^2 = \text{var}[\ln(S_t)|S_{t-1}]$.
- Want market value at $t = 0$ of call option.
- European call option with expiration at time T.
- Payout at T is $\max(S_T - K, 0)$.
- Assume stock does not pay dividends.

The Black-Scholes-Merton formula, contd.

- Three alternative methods lead to same result:
  1. Take limit of binomial model as $n \to \infty, h \to 0$.
  2. Replicating portfolio strategy directly in continuous time.
  3. Find "risk-neutral" expectation of $\max(S_T - K, 0)$; take PV.
- Hull (9th, p. 336) starts on 2, see also exercise 15.17.[1]
- Instead does 3 on pp. 352–353 (9th ed.).[2]
- Hull pp. 298–301 (9th ed.) does 1.[3]
- Today: Will first look at method 2, then 3.
- Result is Black-Scholes-Merton formula,

$$c(S_0, K, T, r, \sigma) \equiv S_0 N(d_1) - Ke^{-rT} N(d_2),$$

where N is the standard normal distribution function, and

$$d_1 \equiv \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad \text{and} \quad d_2 \equiv d_1 - \sigma\sqrt{T}.$$

[1] 8th ed., p. 314, also exerc. 14.17; 7th ed., p. 292, also exerc. 13.17

[2] 8th ed., pp. 329–331; 7th ed., pp. 307–309

[3] 8th ed., pp. 276–279; not in 7th ed.

Portfolio strategies; replicating vs. risk free

- In binomial model, showed a replicating portfolio strategy.
  - Holding Δ shares and B in bonds equals option.
- Hull instead combines share and option to get risk free pf.:
  - Holding Δ shares minus option equals $-B$ bonds.
- In many periods: Need to readjust ...
  - readjust replicating portfolio to replicate option, or
  - readjust risk free portfolio to stay risk free.
- In continuous time: Need to readjust continuously.
- Relies on literal interpretation of "no transaction costs."
- Will show how to determine risk free portfolio strategy.
- This portfolio strategy must earn risk free interest rate.
- If not: Exists riskless arbitrage opportunity.

Risk free portfolio strategy; share and option

(Hull, pp. 331–332 in 9th ed.) (8th ed., pp. 309–310, 7th ed., pp. 287–288)

- S_t is a geometric Brownian motion with drift, an Itô process,

$$dS = \mu S \, dt + \sigma S \, dz.$$

- Before T: Call option value is function of S_t (or S for short).
- Also function of t (or $T - t$, time until expiration).
- What follows is not limited to a call option, $c(S, t)$.
- Valid for any derivative of S, use notation $f(S, t)$.
- Use Itô's lemma for $f(S, t)$,

$$df = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S \, dz.$$

Risk free portfolio strategy, contd.

- For short intervals Δt:

$$\Delta S = \mu S \, \Delta t + \sigma S \, \Delta z.$$

and

$$\Delta f = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \, \Delta z.$$

- Compose portfolio with $\partial f / \partial S$ shares and -1 derivative.
- Value of portfolio is $\Pi = -f + \frac{\partial f}{\partial S} S$.
- Change in value over short Δt is

$$\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S = \left(-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t.$$

- This is risk free since there is no Δz.

Differential equation follows from no arbitrage

- The no-arbitrage condition requires $\Delta\Pi = r\Pi\,\Delta t$.
- This implies
$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r\,f.$$
- This is a partial differential equation (PDE) in $f(S, t)$.
- It has many solutions.
- Only natural, since we have not specified a call option.
- Equation equally valid for put option and other derivatives.

Differential equation, contd.

- To obtain a particular derivative, need *boundary condition:*
- Boundary condition for call option is $f = \max(S - K, 0)$ when $t = T$.
- Black-Scholes-Merton solves PDE and boundary condition.
- Hull leaves this to the reader, exercise 15.17 (9th ed.)[4]
- Technical note: Compare B-S-M formula p. 336 and p. 349.[5]
  - The T on p. 336 is replaced by $T - t$ on p. 349.
  - For a particular option, T is fixed; $T - t$ varies over time.
  - Asking how c varies with time means as $T - t$ goes to zero.
  - t increases until it reaches T.
  - t is the time variable relevant for the partial differential equation.
  - This explains the need for $T - t$ in exercise 15.17.

[4] 8th ed., exerc. 14.17, 7th ed., exerc. 13.17

[5] 8th ed., p. 313 and p. 326, 7th ed., p. 291 and p. 304

Option pricing using "risk-neutral" method

- Based on $\hat{S}_t$, an adjusted process for S_t.
- $\hat{S}_t = S_t e^{(r-\mu)t}$.
- An expected price increase as if investors were risk neutral.
- $E(\hat{S}_T) = S_0 e^{rT}$ instead of $E(S_T) = S_0 e^{\mu T}$.
- ($E(\hat{S}_T)$ is what Hull calls $\hat{E}(S_T)$.)
- (D&D 3rd ed.,[6] p. 34, alternative 3, cf. first lecture, p. 9.)
- Market value at time zero is $e^{-rT} E[\max(0, \hat{S}_T - K)]$.

[6] 2nd ed., p. 25

Using "risk-neutral" method, contd.

- May split the payoff in two parts:
  - Paying K in case $S_T > K$.
  - Receiving S_T in case $S_T > K$.
- Need expectations for each part.
- Instead of S_T, use $\hat{S}_T$, with probability density $g(\hat{S}_T)$:

$$E(K|\hat{S}_T > K) = \int_K^\infty K g(\hat{S}_T) d\hat{S}_T = K \int_K^\infty g(\hat{S}_T) d\hat{S}_T,$$

  which is equal to $K \Pr(\hat{S}_T > K)$; the other part is

$$E(\hat{S}_T|\hat{S}_T > K) = \int_K^\infty \hat{S}_T g(\hat{S}_T) d\hat{S}_T.$$

- The first is simpler, since K is a constant. Next two pages find an expression for this. Then the more complicated $E(\hat{S}_T|\hat{S}_T > K)$.

Valuation of obligation to pay K if $S_T > K$

$$\Pr(\hat{S}_T > K) = \Pr(\ln \hat{S}_T - \ln S_t > \ln K - \ln S_t),$$

where $\ln \hat{S}_T - \ln S_t \sim \phi((r - \sigma^2/2)(T-t), \sigma^2(T-t))$, so that

$$\Pr(\hat{S}_T > K) =$$

$$\Pr\left(\frac{\ln \hat{S}_T - \ln S_t - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} > \frac{\ln K - \ln S_t - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right),$$

where the expression to the left of the inequality sign has a standard normal distribution.

Obligation to pay K if $S_T > K$, contd.

This is thus equal to

$$1 - N\left(\frac{\ln K - \ln S_t - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

The symmetry of the normal distribution means that $1 - N(x) = N(-x)$, so we may rewrite this as:

$$N\left(\frac{\ln S_t - \ln K + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

This means that the valuation of an obligation to pay K if $S_T > K$ is

$$Ke^{-r(T-t)}N\left(\frac{\ln S_t - \ln K + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right),$$

which appears as part of the Black-Scholes-Merton formula.

Valuation of claim to receive S_T if $S_T > K$

Define $h(Q) \equiv \frac{1}{\sqrt{2\pi}} e^{-Q^2/2}$ (a std. normal density), $w \equiv \sigma\sqrt{T-t}$, $m \equiv \ln S_t + (r - \sigma^2/2)(T-t)$, $Q \equiv (\ln \hat{S}_T - m)/w$.

Then Q is a standard normal variable, and can be rewritten as

$$\frac{\ln \hat{S}_T - \ln S_t - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

From the definition of Q we have $\hat{S}_T = e^{wQ+m}$. The conditional expectation we need is

$$E(\hat{S}_T | \hat{S}_T > K) = E\left(e^{wQ+m} \middle| e^{wQ+m} > K\right) = E\left(e^{wQ+m} \middle| Q > \frac{\ln K - m}{w}\right)$$

$$= \int_{\frac{\ln K - m}{w}}^{\infty} e^{wQ+m} h(Q) dQ = \int_{\frac{\ln K - m}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-Q^2/2 + wQ + m} dQ.$$

Claim to receive S_T if $S_T > K$, contd.

The integrand can be rewritten (Hull,[7] 9th ed., p. 353) as:

$$\frac{1}{\sqrt{2\pi}} e^{-(Q^2 - 2wQ + w^2)/2 + m + w^2/2} = e^{m + w^2/2} h(Q - w).$$

(The trick is to observe that a quadratic expression in $(Q - w)$ can be formed by subtracting and adding $w^2/2$. A constant can be factored out, and the new integrand is just another $h()$ expression.) The integral can be rewritten as

$$e^{m+w^2/2} \int_{\frac{\ln K - m}{w}}^{\infty} h(Q - w) dQ = e^{m+w^2/2} \int_{\frac{\ln K - m}{w} - w}^{\infty} h(Y) dY,$$

introducing $Y = Q - w$ as a new variable of integration. Clearly, as Q goes from $(\ln K - m)/w$ to ∞, Y goes from $(\ln K - m)/w - w$ to ∞.

[7] 8th ed., p. 330, 7th ed., p. 308

Claim to receive S_T if $S_T > K$, contd.

The integral with Y is the probability that a standard normal variable exceeds $(\ln K - m)/w - w$. Notice that $e^{m+w^2/2} = S_t e^{r(T-t)}$. Also multiply by $e^{-r(T-t)}$ to get the valuation of the claim,

$$e^{-r(T-t)} E(\hat{S}_T | \hat{S}_T > K) = S_t N \left(\frac{\ln S_t - \ln K + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \right).$$

Conclude: The Black-Scholes-Merton formula

$$c(S_t, K, T-t, r, \sigma) \equiv S_t N(d_1) - Ke^{-r(T-t)} N(d_2),$$

where N is the standard normal distribution function,

$$d_1 \equiv \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad \text{and} \quad d_2 \equiv d_1 - \sigma\sqrt{T-t},$$

(written with S_t and $T-t$ as arguments).

- Together, preceding two slides give the formula.
- Valid for European call options on no-dividend stocks.
- For these, early exercise of American calls is not optimal.
- Thus also valid for American call options on these stocks.
- Or *in periods* when a stock for sure does not pay dividends.

The Black-Scholes-Merton formula, contd.

- Can show that the function $c(S_t, K, T-t, r, \sigma)$ is
  - increasing in S_t,
  - decreasing in K,
  - increasing in $T-t$,
  - increasing in r, and
  - increasing in σ,

  cf. the discussion in lecture 9, pp. 5–6.
- Put option values can be found through put-call parity.
- Formula used a lot in practice; also modified, e.g. for dividends.
- Hull's figs. 11.1–11.2 (9th ed.)[8] show properties of formula (see also pp. 337–338).[9]

[8] 8th ed., figs. 10.1–10.2, 7th ed., figs. 9.1–9.2

[9] 8th ed., p. 315, 7th ed., pp. 292–293

Figure 10.1 Effect of changes in stock price, strike price, and expiration date on option prices when $S_0 = 50$, $K = 50$, $r = 5\%$, $\sigma = 30\%$, and $T = 1$.

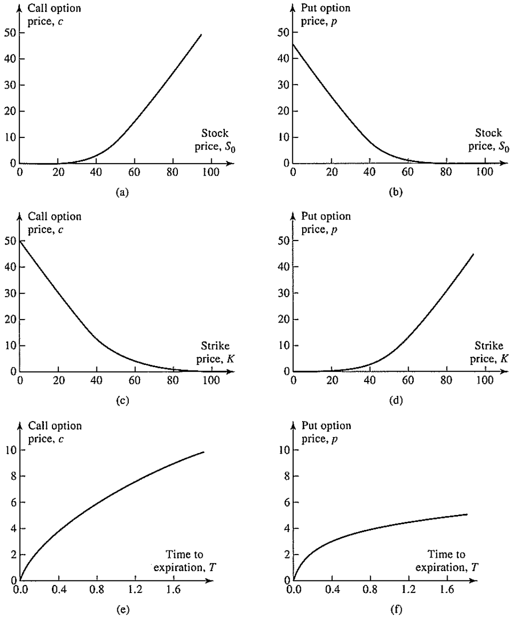
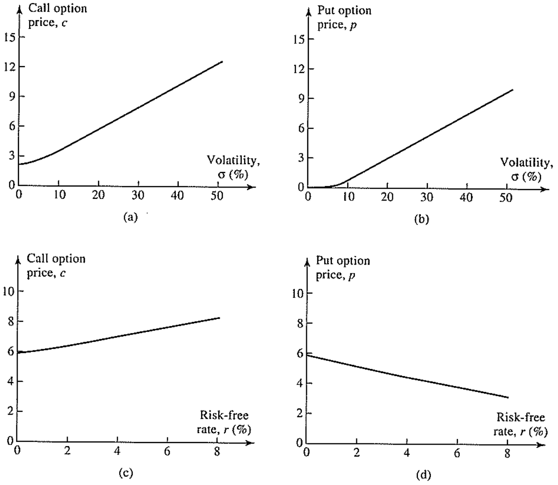


Figure 10.2 Effect of changes in volatility and risk-free interest rate on option prices when $S_0 = 50$, $K = 50$, $r = 5\%$, $\sigma = 30\%$, and $T = 1$.



Dividends in option pricing

- In section 15.12 Hull (9th ed.)[10] considers *known* dividends.
- Dates and magnitudes are both known.
- Much more complicated if one or both are unknown.
- European call: Use $S_t - Y$ instead of S_t.
- Y is present value of dividends to be paid in (t, T).
- Easily understood from risk-neutral valuation method.
- Call option value is $e^{-r(T-t)}E[\max(0, S_T - K)]$, but S_T must be interpreted as the process of the share value without the dividend, which has a starting value of $S_t - Y$ at time t.
- In principle the σ to be used should also reflect this process without dividends (see Hull, footnote 12 in the relevant ch.).

[10] 8th ed., sect. 14.12, 7th ed., sect. 13.12

Dividends in option pricing, contd.

- American call option with dividends: Early exercise?
- Lecture 9, p. 15: If early exercise, then just before dividend.
- Based on this and *known* dividends (Hull, 9th, pp. 344f):[11]
  - Assume the n dividend dates are $t_1 < t_2 < \ldots < t_n < T$.
  - Corresponding dividends are $D_1, \ldots, D_n$.
  - Consider first whether optimal to exercise at t_n.
  - Hull shows: If $D_n \leq K[1 - e^{-r(T-t_n)}]$, never exercise.
  - If $D_n > K[1 - e^{-r(T-t_n)}]$, exercise if S_{t_n} "big enough."
  - Something similar for earlier dividend dates.
  - No exact formulae.
  - Alternative,[12] p. 346: Compare with European options.
  - One with T as expiration, another with t_n.
  - Use the larger of these two European values as approximation.
  - Could maybe extend with more than two dividend dates.

[11] 8th ed., pp. 321f, 7th ed., pp. 299f

[12] 8th ed., pp. 322f, 7th ed., pp. 300f

Volatility, σ

- $\sigma = \sqrt{\text{var}[\ln(S_t/S_{t-1})]}$ is called volatility.
- Only variable in Black-Scholes-Merton not directly observable.
- Must be estimated, typically from time-series data.
- If model is true and constant over time, this is easy.
- If time-varying, may use, e.g., last six months.
- (Perhaps also daily, weekly or monthly data make difference.)
- If models of stock price S_t *and* of option value c_t are true:
- Can compare observed option values with theoretical values.

Implied volatility

- If assume $c_{\text{obs.},t} = c_{\text{theoretical},t} \equiv c(S_t, K, T - t, r, \sigma)$:
- (And assume for sure no dividends are paid until time T:)
- Only one variable, σ, not directly observable in equation.
- May solve equation for σ (cf. Hull,[13] 9th ed., p. 341).
- Called *implicit volatility* or *implied volatility.*
- Solution cannot be found explicitly, but by numerical methods.
- Interpretation: Market uses B-S-M; what σ does it believe?
- Forward-looking number, as opposed to time-series, historical.

[13] 8th ed., p. 318, 7th ed., p. 296

Options and systematic risk

- Until now, no formal link between CAPM and options.
- Option values functions of (S, K, t, r, σ) and perhaps D.
- No direct relation to market portfolio, only through S, σ, r.
- Nevertheless, interesting to ask about option's beta.
- Interesting if option is used for investment.
- Could CAPM and B-S-M models be true simultaneously?
- Need different version of the CAPM, not topic here.

Options and systematic risk, contd.

- But we know a few facts already:
- Replicating portfolio for call option is (Δ, B).
- A positive amount in shares, negative in bonds.
- Invest more than 100 percent of wealth in a share.
- Finance this by borrowing.
- Effect in CAPM: Move northeast in (σ, μ) diagram.
- (Assume share has $\beta > 0$, $E(\tilde{r}) > r_f$.)
- Call option has higher expected rate of return than share.
- Call option thus has higher beta than share.