ECON4510 – Finance Theory
Lecture 7

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Markets for state-contingent claims

(Danthine and Donaldson[1] (3rd) ch. 9 and 11.1–4; Cowell ch. 8.6)

- Theoretically useful framework for markets under uncertainty.
- Used both in simplified versions and in general version, known as *complete markets* (*komplette markeder*) (definition later).
- Extension of standard general equilibrium and welfare theory.
- Developed by Kenneth Arrow and Gerard Debreu during 1950's.
- First and second welfare theorem hold under some assumptions.
- Not necessarily very realistic. Illustrates what assumptions are needed to extend the two welfare theorems to world of uncertainty.

[1] 2nd ed., ch. 8 and 10.1–4

Markets for state-contingent claims, contd.

Description of one-period uncertainty:

- A number of different *states* (*tilstander*) may occur, numbered $\theta = 1, \ldots, N$.
- Here: N is a finite number.
- Exactly one of these will be realized.
- All stochastic variables depend on this state only: As soon as the state has become known, the outcome of all stochastic variables are also known. Any stochastic variable $\tilde{X}$ can then be written as $X(\theta)$.
- "Knowing probability distributions" means knowing each state's (i) probability and (ii) outcomes of stochastic variables.

Securities with known state-contingent outcomes

- Consider M securities (*verdipapirer*) numbered $j = 1, \ldots, M$.
- May think of as shares of stock (*aksjer*).
- Value of one unit of security j will be $p_{j\theta}$ if state θ occurs. These values are known.
- Buying numbers X_j of security j today, for $j = 1, \ldots, M$, will give total outcomes in the N states as follows:

$$\begin{bmatrix} p_{11} & \cdots & p_{M1} \\ \vdots & & \vdots \\ p_{1N} & \cdots & p_{MN} \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ \vdots \\ X_M \end{bmatrix} = \begin{bmatrix} \sum p_{j1} X_j \\ \vdots \\ \sum p_{jN} X_j \end{bmatrix}.$$

- An $N \times 1$ vector with one element for each state.

Known state-contingent outcomes, contd.

- If prices today (period zero) are $p_{10}, \ldots, p_{N0}$, this portfolio costs:

$$[p_{10} \cdots p_{M0}] \cdot \begin{bmatrix} X_1 \\ \vdots \\ X_M \end{bmatrix} = \sum p_{j0} X_j.$$

- Observe that the vector of X's here is *not* a vector of portfolio weights. Instead each X_j is the number of shares (etc.) which is bought of each security.

Constructing a chosen state-contingent vector

If we wish some specific vector of values (in the N states), can any such vector be obtained? Suppose we wish

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}.$$

Can be obtained *if there exist N securities with* **linearly independent** (*lineært uavhengige*) *price vectors,* i.e. vectors

$$\begin{bmatrix} p_{11} \\ \vdots \\ p_{1N} \end{bmatrix}, \cdots, \begin{bmatrix} p_{N1} \\ \vdots \\ p_{NN} \end{bmatrix}.$$

Complete markets

Suppose N such securities exist, numbered $j = 1, \dots, N$, where $N \leq M$. A portfolio of these may obtain the right values:

$$\begin{bmatrix} p_{11} & \cdots & p_{N1} \\ \vdots & & \vdots \\ p_{1N} & \cdots & p_{NN} \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}$$

since we may solve this equation for the portfolio composition

$$\begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{N1} \\ \vdots & & \vdots \\ p_{1N} & \cdots & p_{NN} \end{bmatrix}^{-1} \cdot \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}$$

If there are not as many as N "linearly independent securities," the system cannot be solved in general. If N linearly independent securities exist, the securities market is called *complete.* The solution is likely to have some negative X_j's. Thus short selling must be allowed.

Remarks on complete markets

- To get any realism in description: N must be very large.
- But then, to obtain complete markets, the number of different securities, M, must also be very large.
- Three objections to realism:
  - Knowledge of all state-contingent outcomes.
  - Large number of different securities needed.
  - Security price vectors linearly dependent.
- In an extension to many periods, can show that the necessary numbers of linearly independent securities is equal to the maximal dimension of new information arriving at any point in time. This may be less than the number of states.

Arrow-Debreu securities

- *Securities with the value of one money unit in one state, but zero in all other states.*
- Also called *elementary state-contingent claims*, (*elementære tilstandsbetingede krav*), or *pure securities*.
- Possibly: There exist N different A-D securities.

Arrow-Debreu securities, contd.

- If exist: Linearly independent. Thus complete markets.
- If not exist, but markets are complete: May construct A-D securities from existing securities. For any specific state θ, solve:

$$\begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{N1} \\ \vdots & & \vdots \\ p_{1N} & \cdots & p_{NN} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

with the 1 appearing as element number θ in the column vector on the right-hand side.

State prices

The *state price* for state number θ is the amount you must pay today to obtain one money unit if state θ occurs, but zero otherwise. Solve for state prices:

$$q_\theta = [p_{10} \cdots p_{N0}] \begin{bmatrix} p_{11} & \cdots & p_{N1} \\ \vdots & & \vdots \\ p_{1N} & \cdots & p_{NN} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

State prices are today's prices of A-D securities, if those exist.

Risk-free interest rate

To get one money unit available in *all* possible states, need to buy one of *each* A-D security. Like risk-free bond.

Risk-free interest rate r_f is defined by:

$$\frac{1}{1+r_f} = \sum_{\theta=1}^{N} q_\theta .$$

Pricing and decision making in complete markets

All you need is the state prices. If an asset has state-contingent values

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}$$

then its price today is simply

$$[q_1 \cdots q_N] \cdot \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix} = \sum_{\theta=1}^{N} q_\theta Y_\theta .$$

- Can show this must be true for all traded securities.
- For small potential projects: Also (approximately) true. Exception for large projects which change (all) equilibrium prices.
- Typical investment project: Investment outlay today, uncertain future value. Accept project if outlay less than valuation (by means of state prices) of uncertain future value.

Absence-of-arbitrage proof for pricing rule

If some asset with future value vector

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}$$

is traded for a different price than

$$[q_1 \cdots q_N] \cdot \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix},$$

then one can construct a *riskless arbitrage,* defined as

> A set of transactions which gives us a net gain now, and with certainty no net outflow at any future date.

A riskless arbitrage cannot exist in equilibrium when people have the same beliefs, since if it did, everyone would demand it. (Infinite demand for some securities, infinite supply of others, not equilibrium.)

Proof contd., exploiting the arbitrage

Assume that a claim to

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}$$

is traded for a price

$$p_Y < [q_1 \cdots q_N] \cdot \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}.$$

"Buy the cheaper, sell the more expensive!" Here: Pay p_Y to get claim to Y vector, shortsell A-D securities in amounts $\{Y_1, \ldots, Y_N\}$, cash in a net amount now, equal to

$$[q_1 \cdots q_N] \cdot \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix} - p_Y > 0.$$

Proof contd., exploiting the arbitrage

Whichever state occurs: The Y_θ from the claim you bought is exactly enough to pay off the short sale of a number Y_θ of A-D securities for that state. Thus no net outflow (or inflow) in period one.

Similar proof when opposite inequality.

In both cases: Need short sales.

Value additivity for complete markets (D&D, pp. 335f)

(2nd ed., pp. 204f)

- Assume asset c gives a payoff $\tilde{z}_c = A\tilde{z}_a + B\tilde{z}_b$.
- A, B are constants, $\tilde{z}_a, \tilde{z}_b$ are payoffs for assets a, b.
- Then today's price of c is $p_c = Ap_a + Bp_b$, the same linear combination.
- If not: Riskless arbitrage.
- Also true for CAPM and option pricing models.

Separation principle for complete markets

- As long as firm is small enough — its decisions do not affect market prices — all its owners will agree on how to decide on investment opportunities: Use state prices.
- Everyone agrees, irrespective of preferences and wealth.
- Also irrespective of probability beliefs — may believe in different probabilities for the states to occur.
- Exception: All must believe that the same N states have strictly positive probabilities. (Why?)

Individual utility maximization with complete markets

Assume for simplicity that A-D securities exist. Consider individual who wants consumption today, c_0, and in each state next period, c_θ. Budget constraint (with c_0 as numeraire, i.e., setting the price of c_0 to 1):

$$W_0 = \sum_\theta q_\theta c_\theta + c_0.$$

Let $\pi_\theta \equiv \Pr(\text{state } \theta)$. Assume separable utility function

$$u(c_0) + E[U(c_\theta)].$$

We assume that $U' > 0, U'' < 0$ and similarly for the u function. (Possibly $u() \neq U()$, maybe only because of time preference. Most typical specification is that $U() \equiv \frac{1}{1+\delta} u()$ for some time discount rate δ.)

Individual utility maximization, contd.

$$\max \left[u(c_0) + \sum_\theta \pi_\theta U(c_\theta) \right] \text{ s.t. } W_0 = \sum_\theta q_\theta c_\theta + c_0$$

has f.o.c.

$$\frac{\pi_\theta U'(c_\theta)}{u'(c_0)} = q_\theta \text{ for all } \theta$$

(and the budget constraint).

Remarks on first-order conditions

$$\frac{\pi_\theta U'(c_\theta)}{u'(c_0)} = q_\theta \text{ for all } \theta.$$

Taking $q_1, \dots, q_N$ as exogenous: For any given c_0, consider how to distribute budget across states. Higher $q_\theta \Rightarrow$ higher $U'(c_\theta) \Rightarrow$ lower c_θ. Higher state prices lowers consumption.

Consider now whole securities market. For simplicity consider a pure exchange economy with no productions, so that the total consumption in each future state,

$$\bar{c}_\theta = \sum_{\text{individuals}} c_\theta,$$

is given.

Assume that $\bar{c}_\theta$ increases. Generally people's $U'(c_\theta)$ will decrease. Equilibrium restored through decreasing q_θ. Less scarcity in state θ leads to lower price of consumption in that state.

Remarks on first-order conditions, contd.

Assume now that everyone believes in same $\pi_1, \ldots, \pi_N$. If some π_θ increases, everyone wants own c_θ to increase (as long as state prices q_θ remain fixed). Impossible. Equilibrium restored through higher q_θ.

It is clear that we need an equilibrium model in order to understand how the equilibrium prices depend on exogenous variables (like endowments and preference parameters). There is an example in a seminar exercise, and more in the remaining pages of this lecture.

State contingent claims: Equilibrium and Pareto Optimum

(Danthine & Donaldson, 3rd ed., sections 9.3–9.4)[2]

Simplify as before: Two periods, $t = 0, 1$; N different states of the world may occur at $t = 1$; only one consumption good.

Each consumer, k, derives utility at $t = 0$ from two sources:

- Consumption at $t = 0$, c_0^k.
- Claims to consumption at $t = 1$ in the different states which may occur; this is an N-vector, $(c_{\theta_1}^k, c_{\theta_2}^k, \ldots, c_{\theta_N}^k)$.

Welfare theorems hold in this setup:

- Each time-and-state specified consumption good must be seen as a separate type of good.
- Then the two welfare theorems work just as in a static model without uncertainty.

[2] 2nd ed., sections 8.3–8.4

Equilibrium and Pareto Optimum, contd.

Just as in model without uncertainty:

- Pareto Optimum: Equalities of marginal rates of substitution (MRS).
- Market solution: Consumers equalize MRS's to price ratios, and achieve P.O.
- First welfare theorem: Competitive market solution is Pareto Optimal.
- Second welfare theorem: Any Pareto Optimum can be obtained as a competitive market solution by distributing the initial endowments suitably amongst the consumers.

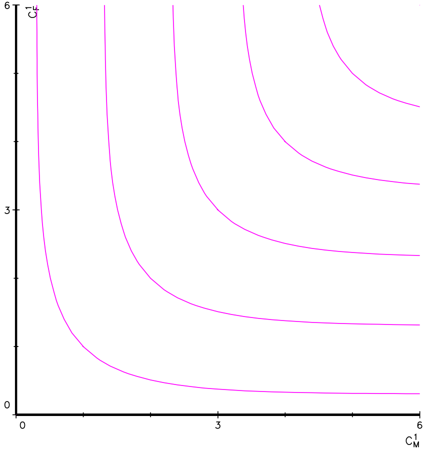
Will look at an example to strengthen the intuitive understanding.

Example: Potato-growers (exam ECON3215/4215 f–2010)

- Two farmers (k=1,2), each growing potatoes, but different fields.
- Derive utility from consumption of potatoes at $t = 1$ only.
- $N = 2$, state 1 called M (mild weather), state 2 F (frost); $\Pr(M) = \pi$.
- Farmer 1: Utility $E[U_1(\tilde{C}_1)]$, output 10 in M, 2 in F.
- Farmer 2: Utility $E[U_2(\tilde{C}_2)]$, output 6 in M, 4 in F.
- Will discuss what is a Pareto Optimum, first-order conditions.
- Specified utility function, $E[-e^{-b_k \tilde{C}_k}]$. (What is b_k?)
- With this utility function, discuss
  - Which allocations are Pareto Optimal? (a) for $b_1 = b_2$, and (b) for $b_1 = 4b_2$.
  - Show that optimum means no trade if $b_2 = 4b_1$.
  - What direction is the trade if $b_2 < 4b_1$, and vice versa? Interpretation?
  - If b_2 is fixed, what happens with the optimum if $b_1 \to 0$?

Indifference curves for farmer 1

- Consumption in state M along horizontal axis, consumption in state F along vertical.
- Indifference curves look similar to those we know from ordinary consumer theory.
- These indifference curves depend on probabilities.



Pareto Optimum in the two-farmer example

Consider first what the problem looks like without specifying the utility function. P.O. is achieved by maximizing expected utility of one farmer for each level of expected utility of the other, given the resource constraint.

$$\max_{C_M^1, C_F^1} \pi U_1(C_M^1) + (1-\pi) U_1(C_F^1)$$

subject to

$$\pi U_2(C_M^2) + (1-\pi) U_2(C_F^2) = \bar{U}_2,$$

and

$$C_M^1 + C_M^2 = 16,$$

and

$$C_F^1 + C_F^2 = 6.$$

The two resource constraints say that the total amount used in state M is 16, the sum of outputs in that state, and similarly for state F.

Pareto Optimum, contd.

There is no consideration here of original ownership of these outputs, or of budget constraints that should be satisfied.

Pareto Optimum could come about by the action of a planner who starts by confiscating the ownership of claims to the outputs, then hands these out to the two farmers.

The first-order conditions for how to hand out will show that this can be done in a variety of ways, along a contract curve in the Edgeworth box.

Pareto Optimum, contd.

The Lagrangian for the maximization problem is:

$$\mathcal{L}(C_M^1, C_F^1, C_M^2, C_F^2) = \pi U_1(C_M^1) + (1-\pi)U_1(C_F^1)$$

$$+\mu[\pi U_2(C_M^2) + (1-\pi)U_2(C_F^2) - \bar{U}_2] + \nu(C_M^1 + C_M^2 - 16) + \xi(C_F^1 + C_F^2 - 6).$$

You can work out the first-order conditions for yourself. They imply:

$$\frac{\pi U_1'(C_M^1)}{(1-\pi)U_1'(C_F^1)} = \frac{\pi U_2'(C_M^2)}{(1-\pi)U_2'(C_F^2)}.$$

The probabilities cancel

$$\frac{U_1'(C_M^1)}{U_1'(C_F^1)} = \frac{U_2'(C_M^2)}{U_2'(C_F^2)}.$$

Pareto Optimum, contd.

We introduce the resource constraints, eliminating C_M^2 and C_F^2:

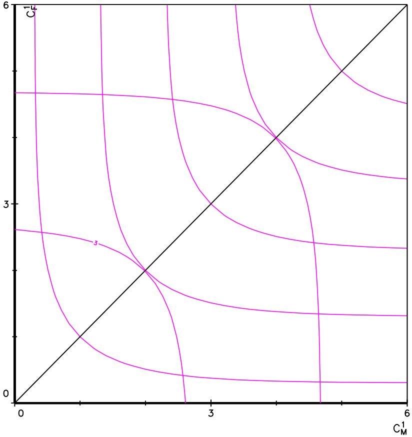
$$\frac{U_1'(C_M^1)}{U_1'(C_F^1)} = \frac{U_2'(16 - C_M^1)}{U_2'(6 - C_F^1)}.$$

The general idea is illustrated in the Edgeworth box on the next slide, although that box has the total output equal to 6 for both states. All points of tangency between the indifference curves of the two farmers are Pareto Optima.

The collection of these points is sometimes called the *contract curve.* If the planner wants a Pareto Optimum, there are many to choose from.

Edgeworth box, quadratic and symmetric

- A case when the two are equally risk averse.
- Length of horizontal side equals total endowment (across farmers) in state M, here set equal to 6.
- Length of vertical, similar for state F, here also 6.
- Each point in box describes one particular distribution of total output between the two, simultaneously for state M and state F.
- In this case, the contract curve becomes diagonal.



Pareto Optimum, contd.

Introduce now $E[U_k(\tilde{C}_k)] \equiv E[-e^{-b_k \tilde{C}_k}]$, with $b_k > 0$ a constant. When the U function is specified like this, we can find a formula for the contract curve and plot it in the Edgeworth box. The first-order condition, equality between MRS's (from slide 29), is now:

$$\frac{b_1 e^{-b_1 C_M^1}}{b_1 e^{-b_1 C_F^1}} = \frac{b_2 e^{-b_2 C_M^2}}{b_2 e^{-b_2 C_F^2}} = \frac{b_2 e^{-b_2 (16 - C_M^1)}}{b_2 e^{-b_2 (6 - C_F^1)}}.$$

This can be solved for

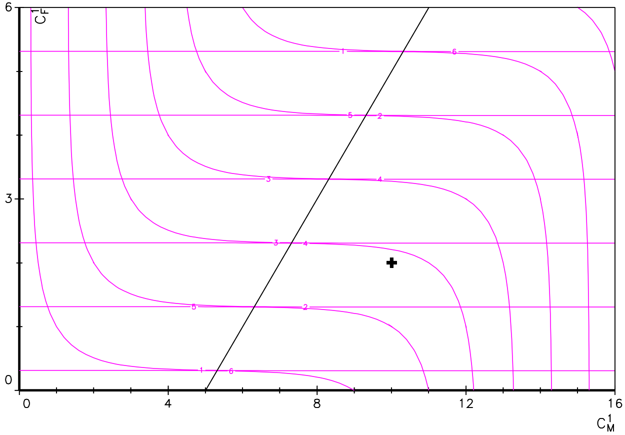
$$-b_1(C_M^1 - C_F^1) = -b_2(16 - C_M^1 - 6 + C_F^1),$$

which gives

$$C_F^1 = C_M^1 - \frac{10}{1 + \frac{b_1}{b_2}}.$$

This is a straight line with slope 45 degrees.

Edgeworth box, rectangular, for the given numbers



If we let $b_1 = b_2$, we find the contract curve:

$$C_F^1 = C_M^1 - \frac{10}{1 + \frac{b_1}{b_2}} = C_M^1 - 5.$$

Pareto Optimum for different values of b_1, b_2

If we let $b_1 = \frac{1}{4} b_2$, we find

$$C_F^1 = C_M^1 - 8,$$

which is a line through the original allocation, $(C_M^1, C_F^1) = (10, 2)$. Thus, for this relationship between the two farmers' aversions to risk, the original allocation was already Pareto Optimal.

With this as a starting point, if b_1 is increased while b_2 is held fixed, the contract curve moves to the left. Farmer 1 is suffering too much from the highly skewed distribution, $C_M^1 > C_F^1$. On the other hand, if $b_1 \to 0$, the contract curve approaces $C_F^1 = C_M^1 - 10$, which means that farmer 2 avoids all risk in the limit.
(This is only a limiting argument. A different utility function would have to be used to write down the model of such a limit, since the function $-e^{b_1 \tilde{C}_1}$ is not a well defined utility function when $b_1 = 0$.)