ECON4510 – Finance Theory
Lecture 5

Kjetil Storesletten
Department of Economics
University of Oslo

March 9, 2017

Stylized example of project valuation

- Suppose project produces two commodities at $t = 1$.
- One variable input is needed at $t = 1$.
- Uncertain prices of input and of both commodities.
- Uncertain quantities of input and of both commodities.
- Net cash flow, $t = 1$: $\tilde{p}_{I1} = \tilde{P}_i \tilde{X}_i + \tilde{P}_j \tilde{X}_j - \tilde{P}_k \tilde{X}_k$.
- For instance, i is milk, j is beef, k is labor.
- (Warning: Many farms are owned by poorly diversified farmers. Then the standard CAPM does not apply.)
- CAPM: $V(\tilde{p}_{I1}) = V(\tilde{P}_i \tilde{X}_i) + V(\tilde{P}_j \tilde{X}_j) - V(\tilde{P}_k \tilde{X}_k)$.
- Four points to be made about this:
  - Flexibility or not?
  - How to value a product of stochastic variables?
  - How to interpret valuation for negative term?
  - How to interpret valuation today of, e.g., beef next year?

Example

$\tilde{p}_{I1} = \tilde{P}_i \tilde{X}_i + \tilde{P}_j \tilde{X}_j - \tilde{P}_k \tilde{X}_k$ at $t = 1$ follows an investment I at $t = 0$.

Flexibility

- Assume that the investment I cannot be cancelled later.
- At $t = 1$, what if one gets outcome $\tilde{p}_{I1} < 0$?
- May assume: Each $\tilde{P}_h$ and $\tilde{X}_h$ always > 0 for $h = i, j, k$.
- Then: $\tilde{p}_{I1} < 0$ happens when $\tilde{P}_k \tilde{X}_k$ is large.
- May be able to cancel project at $t = 1$ if $\tilde{p}_{I1} < 0$.
- If such flexibility, need option valuation methods (study later).
- Then: Value at $t = 1$ will be 0, not $\tilde{p}_{I1}$, when $\tilde{p}_{I1} < 0$.
- Next page assumes no flexibility. Committed to pay $\tilde{P}_k \tilde{X}_k$.
- For some projects, flexibility is realistic. For others, not.

Example, contd.

$\tilde{p}_{I1} = \tilde{P}_i \tilde{X}_i + \tilde{P}_j \tilde{X}_j - \tilde{P}_k \tilde{X}_k.$

Valuation of product of stochastic variables

Quantity uncertainty often local, technical, meteorological (i.e., mainly *idiosyncratic* risk). May simplify valuation of $\tilde{P}\tilde{X}$ expressions if assume: Each $\tilde{X}_h$ ($h = i, j, k$) is stochastically independent of $(\tilde{P}_h, \tilde{r}_M)$. Then: $E(\tilde{P}\tilde{X}) = E(\tilde{P})E(\tilde{X})$ and

$$\mathrm{cov}(\tilde{P}\tilde{X}, \tilde{r}_M) = E(\tilde{P}\tilde{X}\tilde{r}_M) - E(\tilde{P}\tilde{X})E(\tilde{r}_M)$$

$$= E(\tilde{X})\left[E(\tilde{P}\tilde{r}_M) - E(\tilde{P})E(\tilde{r}_M)\right] = E(\tilde{X})\,\mathrm{cov}(\tilde{P}, \tilde{r}_M) \Rightarrow$$

$$V(\tilde{P}\tilde{X}) = E(\tilde{X})V(\tilde{P}), \text{ quantity uncertainty irrelevant.}$$

Example, contd.

$\tilde{p}_{I1} = \tilde{P}_i\tilde{X}_i + \tilde{P}_j\tilde{X}_j - \tilde{P}_k\tilde{X}_k.$

Valuation of negative term

$$V(-\tilde{P}_k\tilde{X}_k) = -E(\tilde{X}_k) \cdot \frac{1}{1+r_f}\left[E(\tilde{P}_k) - \lambda\,\mathrm{cov}(\tilde{P}_k, \tilde{r}_M)\right].$$

- If the covariance increases, then value *increases.*
- High covariance between input price and $\tilde{r}_M$ is good.
- Reason: Project owners are committed to the expense.
- Prefer
  - expense is high when they are otherwise wealthier,
  - expense is low when they are otherwise poorer.

Example, contd.

$\tilde{p}_{I1} = \tilde{P}_i \tilde{X}_i + \tilde{P}_j \tilde{X}_j - \tilde{P}_k \tilde{X}_k.$

Valuation at $t = 0$ of claim to commodity at $t = 1$

- Might perhaps calculate $V(\tilde{P}_j)$ from time series estimates of $E(\tilde{P}_j)$ and $\text{cov}(\tilde{P}_j, \tilde{r}_M)$.
- "Value today of receiving one unit of beef next period."
- In general *not* equal to price of beef today.
- Would have equality if beef were investment object, like gold.
- Instead $V(\tilde{P}_j)$ is present value of *forward price* of beef.
- Usually lower than price of beef today.
- More about this later in course.

Application: Borrowing to finance part of investment

- Consider a firm that raises aI from shareholders and $(1-a)I$ as a loan to finance real investment I, with $a \in (0,1)$.
- Assume investment results in a cash flow $\tilde{p}_I$ one period later.
- Loan will be paid back with interest, a total of $(1-a)(1+r_f)I$.
- For simplicity: Assume now that payback will happen with certainty, irrespective of the outcome of $\tilde{p}_I$.
- This is called a risk free loan, which is not the typical case in practice.
- (Typical case: There is some chance of default (*konkurs*), in which case the loan is not paid back in full. Ignore this for now.)
  - The loan may be risk free if $\Pr(\tilde{p}_I > (1-a)(1+r_f)I) = 1$, which may occur because the uncertainty about the outcome of $\tilde{p}_I$ is not too big, and $(1-a)$ is not too big.
  - The loan could also be risk free because the shareholders somehow guarantee to pay back, by putting up collateral, in addition to $\tilde{p}_I$.
  - We assume the loan must be risk free for the lender to be satisfied with r_f as the promised interest rate.

Borrowing: Consequences for investment decision

- Value today of shareholder's claim is:

$$
\begin{aligned}
p_0 = V(\tilde{p}_I - (1-a)(1+r_f)I) &= V(\tilde{p}_I) - V((1-a)(1+r_f)I) \\
&= \frac{1}{1+r_f}\left[E(\tilde{p}_I) - \lambda \operatorname{cov}(\tilde{p}_I, \tilde{r}_M)\right] - (1-a)I.
\end{aligned}
$$

- Investment decision depends on whether $p_0 > aI$. Simplify to:

$$
\frac{1}{1+r_f}\left[E(\tilde{p}_I) - \lambda \operatorname{cov}(\tilde{p}_I, \tilde{r}_M)\right] > I.
$$

- There is no a in this inequality.
- The investment decision does not depend on the loan financing.
- A version of Miller-Modigliani's capital structure irrelevance theorem

Borrowing: Consequences for beta of shares

- When correctly priced at p_0, the beta of the shares is

$$\frac{1}{\sigma_M^2}\,\mathrm{cov}\left[\frac{\tilde{p}_I-(1-a)(1+r_f)I}{p_0},\tilde{r}_M\right]$$

$$=\frac{V(\tilde{p}_I)}{V(\tilde{p}_I)-(1-a)I}\cdot\frac{1}{\sigma_M^2}\,\mathrm{cov}\left[\frac{\tilde{p}_I}{V(\tilde{p}_I)},\tilde{r}_M\right].$$

- When, also, project is exactly marginal, $V(\tilde{p}_I)=I$, the beta is

$$\frac{1}{a}\cdot\frac{1}{\sigma_M^2}\,\mathrm{cov}\left[\frac{\tilde{p}_I}{V(\tilde{p}_I)},\tilde{r}_M\right]=\frac{1}{a}\cdot\beta_{\text{(without borrowing)}}$$

  higher, the more is borrowed, i.e., the higher is $(1-a)$ and $\frac{1}{a}$.
- For a non-marginal project, a higher $V(\tilde{p}_I)/I$ will reduce beta.

CAPM: Some remarks on realism and testing

- CAPM equation can be tested on time-series data.
- Need r_f, need $\tilde{r}_M$, need stability.

Existence of risk free rate

- Interest rates on government bonds are nominally risk free.
- (Exception: Exist CPI-indexed bonds in some countries.)
- With inflation: Real interest rates are uncertain.
- Real rates of return are what agents really care about.
- Alternative model: No risk free rate. D&D sect. 8.4–8.7.[1]
- Without r_f, still CAPM equation with testable implications.

[1] Sect. 7.4–7.7 in 2nd ed.

CAPM: More remarks on realism and testing

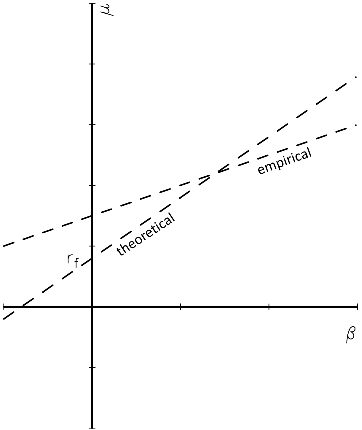
Stability of expectations, variances, covariances

- CAPM says nothing testable about single outcome.
- Need repeated outcomes, i.e., time series.
- Outcomes must be from probability distribution.
- Requires stability over some time.
- A problem, perhaps not too bad.

CAPM: More remarks on realism and testing, contd.

- Empirical SML often has too high intercept, too low slope.
- Can find other significant variables:
  - Asset-specific variables in cross-section.
  - Economy-wide variables in time series.

  If these are pre-determined at $t = 0$: Conditional CAPM.



Closer look at CAPM—the need for an equilibrium model

- What will be effect of merging two firms?
- What will be effect of a higher interest rate?
- Could interest rate r_f exceed $E(\tilde{r}_{mvpf})$ (min-variance-pf)?
- What will be effect of taxation?

Need equilibrium model to answer this. Partial equilibrium: Consider stock market only. Typical competitive partial equilibrium model:

- Specify demand side: Who? Preferences?
- Leads to demand function.
- Specify supply side: Who? Preferences?
- Leads to supply function.
- Each agent views prices as exogenous.
- Supply = demand gives equilibrium, determines prices.

The need for an equilibrium model, contd.

Repeating assumptions so far:

- Two points in time, beginning and end of period, $t = 0, 1$.
- Competitive markets. No taxes or transaction costs.
- All assets perfectly divisible. Short sales are allowed.
- Agent h has exogenously given wealth W_0^h at $t = 0$.
- Wealth at $t = 1$, $\tilde{W}^h$, is value of portfolio composed at $t = 0$.
- Agent h risk averse, cares only about mean and var. of $\tilde{W}^h$.
- Portfolio composed of one risk free and many risky assets.
- Agents view r_f as exogenous.
- Agents view probability distn. of $\tilde{r}_j$ vector as exogenous.
- All believe in the same probability distributions.

Main results:

- CAPM equation, $\tilde{r}_j = r_f + \beta_j[E(\tilde{r}_m) - r_f]$.
- Everyone composes risky part of portfolio in the same way.

Partial equilibrium model of stock market

Maintain all previous assumptions. Add these:

- The number of agents is H, $h = 1, \dots, H$.
- The number of different assets is $n + 1$, $j = 0, \dots, n$.
- Before trading at $t = 0$, all assets owned by the agents: $\bar{X}_j^h$.
- After trading at $t = 0$, all assets owned by the agents: X_j^h.
- Agents own nothing else, receive no other income.
- Asset values at $t = 1$, $\tilde{p}_{j1}$, exogenous probability distribution.
- One of these is risk free.
- Asset values at $t = 0$, p_{j0}, endogenous for $j = 1, \dots, n$.
- But each agent views the p_{j0}'s as exogenous.
- Thus each agent views prob. distn. of $\tilde{r}_j \equiv \tilde{p}_{j1}/p_{j0} - 1$ as exogenous.
- W_0^i consists of asset holdings, $W_0^i = \sum_{j=1}^n p_{j0} \bar{X}_j^i$.
- Thus each agent views own wealth, W_0^i, as exogenous.

Interpretation of model setup

- Pure exchange model. No production. No money.
- Utility attached to asset holdings after trade at $t = 0$.
- Market at $t = 0$ allows for reallocation of these.
- Pareto improvement: Agents trade only what they want.
- At $t = 1$ no trade, only payout of firms' realized values.

Equilibrium response to increased risk free rate?

- Previous results:

$$p_{j0} = \frac{1}{1+r_f}\left[E(\tilde{p}_{j1}) - \lambda \operatorname{cov}(\tilde{p}_{j1}, \tilde{r}_M)\right], \text{ with } \lambda = \frac{E(\tilde{r}_M) - r_f}{\operatorname{var}(\tilde{r}_M)},$$

$$E(\tilde{r}_j) = r_f + \frac{\operatorname{cov}(\tilde{r}_j, \tilde{r}_M)}{\operatorname{var}(\tilde{r}_M)}\left[E(\tilde{r}_M) - r_f\right].$$

- None of these have only exogenous variables on right-hand side.
- In both, $\tilde{r}_M$ on right-hand side is endogenous in model.
- Consider hyperbola and tangency in σ, μ diagram:
  - If r_f is increased, tangency point seems to move up and right.
  - Increase in $E(\tilde{r}_M)$ seems to be less than increase in r_f, and $\operatorname{var}(\tilde{r}_M)$ is increased; question, does this imply $E(\tilde{r}_j)$ increases (but less than ...)?
  - Argument too simple; relies on keeping hyperbola fixed.
  - CAPM equation shows that $E(\tilde{r}_j)$ is likely to change.
  - True for all risky assets, thus entire hyperbola changes.
- To detect effect of Δr_f, need only exogenous variables on RHS.
- Not part of this course.

D&D, sect. 8.4–8.7: CAPM without risk free asset

Sect. 7.4–7.7 in 2nd ed.

Main points:

- Consider N risky assets, $N > 2$, no risk free asset. Then the frontier portfolio set is an hyperbola. (Mentioned without proof in third lecture.)
- Can derive version of CAPM without risk free asset. Important if, e.g., there is uncertain inflation.

The version mentioned in the second point is important to understand much of the CAPM literature. Market portfolio plays important role also in that version of the model, even though it is *not* equal to the risky part of everyone's portfolio.

Will relate to parts of the discussion in D&D, and will use their equation numbers.

But will simplify, and skip some of their intermediate results which are not necessary for the main results we need.

D&D, sect. 8.4–8.7: CAPM without risk free asset

Sect. 7.4–7.7 in 2nd ed.

The new version of the CAPM can be illustrated in the σ, μ diagram we have used previously. Following D&D, the expected rate of return for portfolio p will now be denoted E_p, not μ_p, but this is the same variable.

Observe that Figure 8.6 on p. 225 of D&D[2] is a different diagram. It has σ^2 on the horizontal axis, not σ. The frontier portfolio set is thus a parabola, not an hyperbola, in that diagram.

The particular geometrical properties shown there, will not exist in a σ_p, E_p diagram. We will skip Figure 8.6 in the discussion which follows.

[2] Fig. 7.6 p. 134 in 2nd ed.

Differentiation of vectors and matrices

- Def.: Derivative of a (scalar) function with respect to an $N \times 1$ vector is the $N \times 1$ vector of derivatives with resp. to each element.
- Def.: Derivative of a (scalar) function with respect to an $1 \times N$ vector is the $1 \times N$ vector of derivatives with resp. to each element.

$$x = (x_1, \ldots, x_N) \Longrightarrow \frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N} \right).$$

- $\Rightarrow$ Derivative of scalar product: a and x both are $N \times 1$:

$$\frac{\partial}{\partial x}(a^T \cdot x) = a^T \quad \text{(where } ^T \text{ denotes transpose).}$$

(Generalizes well known scalar result, $\partial(b \cdot y)/\partial y = b$.)

- Def.: Derivative of $M \times 1$ vector w.r.t. $N \times 1$ vector is $M \times N$ matrix of derivatives.

Differentiation of vectors and matrices, contd.

- $\Rightarrow$ Derivative of matrix-vector product: A is $M \times N$, x is $N \times 1$:

$$\frac{\partial}{\partial x}(Ax) = A.$$

- $\Rightarrow$ Derivative of quadratic form: A is $N \times N$, x is $N \times 1$:

$$\frac{\partial}{\partial x}(x^T A x) = x^T (A + A^T).$$

  (Generalizes scalar $\partial(b \cdot y^2)/\partial y = 2b \cdot y$.)

- $\Rightarrow$ Symmetric version of the same: $A = A^T$ is $N \times N$ symmetric:

$$\frac{\partial}{\partial x}(x^T A x) = 2x^T A.$$

- Def.: Boldface $\mathbf{1}$ denotes vector of ones: $\mathbf{1} = (1, \ldots, 1)^T$.