ECON4510 – Finance Theory
Lecture 4

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Illustrating the Security Market Line

- All securities are located on the line, $\mu_j = r_f + \beta_j(\mu_M - r_f)$.
- Also any portfolio of m securities. Show for $m = 2$:

$$\mu_p = a\mu_i + (1-a)\mu_j = a[r_f + \beta_i(\mu_M - r_f)] + (1-a)[r_f + \beta_j(\mu_M - r_f)]$$
$$= r_f + [a\beta_i + (1-a)\beta_j](\mu_M - r_f)$$

$$= r_f + \left[a\frac{\text{cov}(\tilde{r}_i, \tilde{r}_M)}{\sigma_M^2} + (1-a)\frac{\text{cov}(\tilde{r}_j, \tilde{r}_M)}{\sigma_M^2}\right](\mu_M - r_f)$$
$$= r_f + \frac{\text{cov}(a\tilde{r}_i + (1-a)\tilde{r}_j, \tilde{r}_M)}{\sigma_M^2}(\mu_M - r_f)$$
$$= r_f + \frac{\text{cov}(\tilde{r}_p, \tilde{r}_M)}{\sigma_M^2}(\mu_M - r_f) = r_f + \beta_p(\mu_M - r_f).$$

- We have found that $\beta_p = a\beta_i + (1-a)\beta_j$, a value-weighted average of β_i and β_j.

Interpretation of CAPM equation

$$
\begin{aligned}
E(\tilde{r}_j) &= \mu_j = r_f + \beta_j(\mu_M - r_f) = r_f + \frac{\sigma_{jM}}{\sigma_M^2}(E(\tilde{r}_M) - r_f) \\
&= r_f + \sigma_j \cdot corr(\tilde{r}_j, \tilde{r}_M) \cdot \underbrace{\frac{E(\tilde{r}_M) - r_f}{\sigma_M}}_{\text{Sharpe ratio}}
\end{aligned}
$$

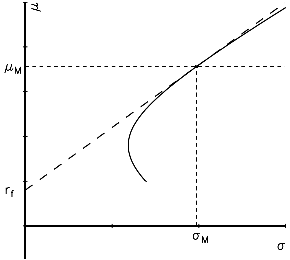
Verbal interpretation:

> *The expected rate of return on any asset depends on only one characteristic of that asset, namely its rate of return's covariance with the rate of return on the market portfolio.*
>
> *The expected rate of return is equal to the risk free interest rate plus a term which depends on a measure of risk. (Higher risk means higher expected rate of return.) The relevant measure of risk is the asset's beta. This is multiplied with the expected excess rate of return on the market portfolio.*

Interpretation of CAPM equation, contd.

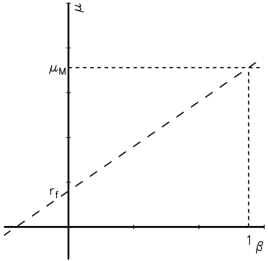
- Obvious: Does not say what realization of r_j will be. Only $E(\tilde{r}_j)$.
- Risk measure depends on covariance because the covariance determines how much that asset will contribute to the risk of the agent's portfolio.
- This is true for any agent, since all hold the same risky portfolio.
- May have $\text{cov}(\tilde{r}_j, \tilde{r}_M) < 0$, so that $\beta_j < 0$.
  - Not very common in practice.
  - Such assets contribute to reducing σ_M (when included in M).
  - Valuable to investors. (Question: What does this imply for value at $t = 0$? For expected rate of return?)

Avoid confusion between these two diagrams:



The *Capital Market Line:*

- Efficient set given 1 risk free and *n* risky assets.
- Relevant for choice between alternative portfolios.
- Drawn in (σ_p, μ_p) diagram.
- A ray (half-line) starting at $(0, r_f)$ in that diagram.



The *Security Market Line:*

- Location of *all* traded assets in equilibrium.
- Also location of any portfolio of these assets.
- Drawn in (β_j, μ_j) diagram.
- A line through $(0, r_f)$ in that diagram.

Other forms of the CAPM equation

Remember: $1 + \tilde{r}_j = \tilde{p}_{j1}/p_{j0}$. Rewrite:

$$\frac{E(\tilde{p}_{j1})}{p_{j0}} - 1 = r_f + \frac{\mu_M - r_f}{\sigma_M^2} \operatorname{cov}\left(\frac{\tilde{p}_{j1}}{p_{j0}} - 1, \tilde{r}_M\right)$$

$$= r_f + \frac{\mu_M - r_f}{\sigma_M^2}\left[\operatorname{cov}\left(\frac{\tilde{p}_{j1}}{p_{j0}}, \tilde{r}_M\right) - \operatorname{cov}(1, \tilde{r}_M)\right]$$

$$\iff \frac{E(\tilde{p}_{j1})}{p_{j0}} = 1 + r_f + \lambda \operatorname{cov}\left(\frac{\tilde{p}_{j1}}{p_{j0}}, \tilde{r}_M\right)$$

$$= 1 + r_f + \lambda \operatorname{cov}\left(\frac{1}{p_{j0}}\tilde{p}_{j1}, \tilde{r}_M\right),$$

with λ defined as $(\mu_M - r_f)/\sigma_M^2$.

$\frac{1}{p_{j0}}$ is deterministic, and can be factored out of the covariance.

Other forms of the CAPM equation, contd.

$$\iff \quad \frac{E(\tilde{p}_{j1})}{p_{j0}} = 1 + r_f + \frac{\lambda \operatorname{cov}(\tilde{p}_{j1}, \tilde{r}_M)}{p_{j0}}.$$

Multiply both sides by

$$\frac{p_{j0}}{1 + r_f}$$

and rearrange terms to get:

$$p_{j0} = \frac{1}{1 + r_f} \left[E(\tilde{p_{j1}}) - \lambda \operatorname{cov}(\tilde{p}_{j1}, \tilde{r}_M) \right].$$

Expression in square bracket is called the *certainty equivalent* (in the CAPM sense) of $\tilde{p}_{j1}$. Price today is present value of expression, just as if that would be received with certainty one period into the future.

Not what is called certainty equivalent in expected utility theory.

Other forms of the CAPM equation, contd.

Can also rewrite as

$$p_{j0} = \frac{E(\tilde{p}_{j1})}{1 + r_f + \lambda \operatorname{cov}(\tilde{r}_j, \tilde{r}_M)}.$$

(Prove yourself how to arrive at this!)

- Like previous form: p_{j0} on left-hand side.
- But here: Right-hand side is "expected present value."
- Denominator: one plus *risk-adjusted discount rate,* RADR.
- Obtained by adding something to r_f.
- Risk-adjustment again depends on λ and covariance.
- Observe difference in covariance expressions!
- Previously: $\operatorname{cov}(\tilde{p}_{j1}, \tilde{r}_M)$.
- Now: $\operatorname{cov}(\tilde{r}_j, \tilde{r}_M) \equiv \operatorname{cov}(\tilde{p}_{j1}/p_{j0}, \tilde{r}_M)$.
- If need to solve for p_{j0}, the formula on top of this page is less useful: It has p_{j0} also on right-hand side, "hidden" in covariance term.
- Use instead expression on previous page.

CAPM: Implications, applications

- For a portfolio: Variance the relevant risk measure.
- But for each security: Covariance with $\tilde{r}_M$ is the relevant risk measure
- Reason: Covariance measures contribution to portfolio variance.
- Generally: Covariance with each agent's marginal utility.
- But when mean-var preferences: Covariance with agent's wealth.
- In equilibrium: All have same risky portfolio (composition).
- Thus covariance with $\tilde{r}_M$ relevant for everyone.
- Implication: Adding noise does not matter.

CAPM: Implications, applications, contd.

- Suppose $\tilde{p}_{h1} = \tilde{p}_{j1} + \tilde{\varepsilon}$, where $E(\tilde{\varepsilon}) = 0$ and $\tilde{\varepsilon}$ is stochastically independent of $(\tilde{p}_{j1}, \tilde{r}_M)$. Can show $p_{h0} = p_{j0}$:

$$p_{h0} = \frac{1}{1+r_f}\left[E(\tilde{p}_{j1} + \tilde{\varepsilon}) - \lambda \operatorname{cov}(\tilde{p}_{j1} + \tilde{\varepsilon}, \tilde{r}_M)\right]$$

$$= \frac{1}{1+r_f}\left[E(\tilde{p}_{j1}) + E(\tilde{\varepsilon}) - \lambda[\operatorname{cov}(\tilde{p}_{j1}, \tilde{r}_M) + \operatorname{cov}(\tilde{\varepsilon}, \tilde{r}_M)]\right] = p_{j0},$$

since both terms containing $\tilde{\varepsilon}$ are zero.

- At this point: Do *not* get confused by ρ_{jM}.
- Common confusion: Since $\sigma_{jM} = \rho_{jM}\sigma_j\sigma_M$, some believe that a higher σ_j leads to a higher σ_{jM}, as if $\sigma_h > \sigma_j \Rightarrow \sigma_{hM} > \sigma_{jM}$.
- Top of page shows a case where this is not true; instead:

$$\sigma_{hM} = \sigma_{jM} \text{ and } \rho_{hM} = \frac{\sigma_{hM}}{\sigma_h\sigma_M} = \frac{\sigma_{jM}}{\sigma_h\sigma_M} < \frac{\sigma_{jM}}{\sigma_j\sigma_M} = \rho_{jM}.$$

CAPM terminology: Systematic vs. unsystematic risk

Define $\tilde{\eta}_j$ through the equation

$$\tilde{r}_j = r_f + \beta_j \left[\tilde{r}_M - r_f \right] + \tilde{\eta}_j.$$

Then the CAPM equation

$$E(\tilde{r}_j) = r_f + \beta_j \left[E(\tilde{r}_M) - r_f \right]$$

implies that $E(\tilde{\eta}_j) = 0$, and that:

$$\text{cov}(\tilde{\eta}_j, \tilde{r}_M) = \text{cov}(\tilde{r}_j, \tilde{r}_M) - \beta_j \, \text{var}(\tilde{r}_M) = \sigma_{jM} - \beta_j \sigma_M^2 = 0.$$

This allows us to split σ_j^2 in two parts:

$$\text{var}(\tilde{r}_j) \equiv \sigma_j^2 = \beta_j^2 \sigma_M^2 + \sigma_{\eta_j}^2 = \beta_j \sigma_{jM} + \sigma_{\eta_j}^2.$$

Systematic vs. unsystematic risk, contd.

Repeat:

$$\text{var}(\tilde{r}_j) \equiv \sigma_j^2 = \beta_j^2 \sigma_M^2 + \sigma_{\eta_j}^2 = \beta_j \sigma_{jM} + \sigma_{\eta_j}^2.$$

First term called *systematic risk.* This is reflected in the market valuation. (More general term: *Relevant risk* or *covariance risk.*)

Second term called *unsystematic risk.* As we have seen, it is not reflected in market valuation. (More general term: *Irrelevant risk.*)

The sum of the two called *total risk* or *variance risk*. This *is* relevant for portfolios, evaluated for being the total wealth of someone, but not for individual securities, to be combined with other securities in portfolios.

Valuation of firms, value additivity

$$p_{i0} = \frac{1}{1+r_f}\left[E(\tilde{p}_{i1}) - \lambda\,\text{cov}(\tilde{p}_{i1}, \tilde{r}_M)\right] \equiv V(\tilde{p}_{i1}),$$

defines valuation function $V()$, given some $r_f, \tilde{r}_M$.

- $\tilde{p}_{i1}$ is value of share at $t = 1$, including dividends.
- Consider firm financed 100% by equity (no debt).
- Firm's net cash flow at $t = 1$ goes to shareholders.
- Plug in that net cash flow instead of $\tilde{p}_{i1}$.
- Then formula gives value of *all shares* in firm.
- What if $\tilde{p}_{i1} = a\tilde{p}_{j1} + b\tilde{p}_{k1}$? ($a > 0,\ b > 0$ constants.)

Valuation of firms, value additivity, contd.

- Will show that linearity of $E()$ and cov() implies $p_{i0} = ap_{j0} + bp_{k0}$:

$$
\begin{aligned}
p_{i0} &= \frac{1}{1+r_f}\left[E(\tilde{p}_{i1}) - \lambda \operatorname{cov}(\tilde{p}_{i1}, \tilde{r}_M)\right] \\
&= \frac{1}{1+r_f}\left[E(a\tilde{p}_{j1} + b\tilde{p}_{k1}) - \lambda \operatorname{cov}(a\tilde{p}_{j1} + b\tilde{p}_{k1}, \tilde{r}_M)\right] \\
&= \frac{1}{1+r_f}\left[aE(\tilde{p}_{j1}) + bE(\tilde{p}_{k1}) - \lambda[a \operatorname{cov}(\tilde{p}_{j1}, \tilde{r}_M) + b \operatorname{cov}(\tilde{p}_{k1}, \tilde{r}_M)]\right] \\
&= ap_{j0} + bp_{k0}.
\end{aligned}
$$

- Diversification is no justification for mergers.
- Diversification can be done by shareholders.

β of sum is value-weighted average of βs of parts

- Expand result from today's p. 2: Find β of claim to $a\tilde{X} + b\tilde{Y}$.
- Two elements with $\beta_X = \frac{\text{cov}(\tilde{X}/V(\tilde{X}), \tilde{r}_m)}{\sigma_m^2}$ and $\beta_Y = \frac{\text{cov}(\tilde{Y}/V(\tilde{Y}), \tilde{r}_m)}{\sigma_m^2}$.
- "β of the sum" characterizes the return $\frac{a\tilde{X}+b\tilde{Y}}{V(a\tilde{X}+b\tilde{Y})}$ (minus 1?).
- Rewrite return as $\frac{aV(\tilde{X})}{V(a\tilde{X}+b\tilde{Y})} \cdot \frac{\tilde{X}}{V(\tilde{X})} + \frac{bV(\tilde{Y})}{V(a\tilde{X}+b\tilde{Y})} \cdot \frac{\tilde{Y}}{V(\tilde{Y})}$.
- Define the value weights $w = \frac{aV(\tilde{X})}{V(a\tilde{X}+b\tilde{Y})}$ and $1 - w$ ....
- These are non-stochastic $\Rightarrow \text{cov}((w\frac{\tilde{X}}{V(\tilde{X})} + (1-w)\frac{\tilde{Y}}{V(\tilde{Y})}), \tilde{r}_m)$
  $= w\,\text{cov}(\frac{\tilde{X}}{V(\tilde{X})}, \tilde{r}_m) + (1-w)\,\text{cov}(\frac{\tilde{Y}}{V(\tilde{Y})}, \tilde{r}_m)$.
- Divide through by σ_m^2, and find that the β of a claim to a sum (or any linear combination) of cash flows is the value-weighted sum of the βs of each cash flow. Can be expanded to more than two elements.

Investment project's rate of return

- Consider (potential) real investment project:
  - Outlay I at $t = 0$.
  - Revenue $\tilde{p}_{I1}$ at $t = 1$.
- Project value? Should project be undertaken?
- Assume 100% equity financed. (Assume separate firm?)
- Project's rate of return is $(\tilde{p}_{I1} - I)/I$.
- If use of restricted technology or resources: No reason for SML equation to hold for this rate of return.
- May earn "above-normal" expected rate of return.

Investment project's rate of return, contd.

- However: Valuation of $\tilde{p}_{I1}$ possible:

$$p_{I0} = V(\tilde{p}_{I1}) = \frac{1}{1+r_f}\left[E(\tilde{p}_{I1}) - \lambda \operatorname{cov}(\tilde{p}_{I1}, \tilde{r}_M)\right]$$

  defines p_{I0} *independently of* I.
- If $p_{I0} > I$, undertake project. Net value $p_{I0} - I$.
- If $p_{I0} < I$, drop project. Net value of opportunity is 0. Net value of *having to* undertake project is $p_{I0} - I$, negative.
- If perfect competition and free entry for this type of project $\Rightarrow \quad p_{I0} = I$ in long run, through increased supply, lower $E(\tilde{p}_{I1})$.
  - Possible reasons why one could have $p_{I0} \neq I$ in long run: financial frictions such as liquidity premium (project is diffiult to liquidate), not free entry (e.g., only a small set of investors are allowed to purchase a part of the project)

Project valuation

- If claim to $\tilde{p}_{I1}$ costs (formula) p_{I0}, this is equilibrium price.
- Project is then on security market line (SML).
- If project available at different cost I: Not at SML.

The equilibrium ratio $E(\frac{\tilde{p}_{I1}}{p_{I0}})$ corresponds to β_k in

$$E(\frac{\tilde{p}_{I1}}{p_{I0}}) = 1 + r_f + [\mu_M - r_f]\beta_k$$

$$= 1 + r_f + [\mu_M - r_f]\,\text{cov}(\frac{\tilde{p}_{I1}}{p_{I0}}, \tilde{r}_M)/\sigma_M^2.$$

Expressed in terms of exogenous variables (eliminating p_{I0}) this becomes:

$$\beta_k = \frac{(1 + r_f)}{\sigma_M^2 \frac{E(\tilde{p}_{I1})}{\text{cov}(\tilde{p}_{I1}, \tilde{r}_M)} - \mu_M + r_f},$$

independent of I. Only if $I = p_{I0} = V(\tilde{p}_{I1})$, will the project rate of return $\tilde{p}_{I1}/I - 1$ satisfy the CAPM equation with β_k.