Problem set 3: Portfolio choice and oppotunity sets

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Exercise 1: Portfolio choice with two risky assets

An investor has access to two different risky assets. The first asset has expected return $E(r_1) = \mu_1$ and variance $Var(r_1) = \sigma_1^2$. The second has expected return $E(r_2) = \mu_2$ and variance $var(r_2) = \sigma_2^2$. Their covariance is $cov(r_1, r_2) = \sigma_{12} > 0$.

Note that we are not discussing the investors demand for assets or his or her preferences over mean and variance, only the availability of different portfolios.

(a) Write down the return on the portfolio when investing a in asset 1 and $1 - a$ in asset 2. Find the mean μ_p of the investors portfolio.

Hint: there should not be any wealth variable present in your answer. Think of the portfolio as an allocation rule for one unit of wealth.

(b) What is the variance σ_p^2 of this portfolio for some given a? Find the a that minimizes this variance.

(c) Assume that $\sigma_1 < \sigma_2$ and that $-1 < \rho_{12} < 1$. As an intermediary step, show that this implies that $\sigma_1^2 + \sigma_2^2 - 2\sigma_{12} > 0$. Next, show that a_{min} in this setting must be larger than 0.5.

(d) Assume that asset 2 has both lower mean return and higher variance than asset 1. Why might an investor who cares about both mean and variance still want to have holdings of both assets?

Hint: The central argument here can be shown mathematically, have a look at appendix 6.2 in Danthine and Donaldson (the *subsection about imperfectly correlated assets), but I really just want some intuition so there is no need for math.*

(e) Recall that the correlation between two assets is given by $\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$, where $\sigma_i = \sqrt{\sigma_i^2}$. Write μ_p as a function of σ_p (the standard deviation, not the variance) when

(i) The assets are perfectly correlated: $\rho_{12} = 1$, and

(ii) the assets are perfectly negatively correlated: $\rho_{12} = -1$.

Draw both cases in a sigma-mu diagram. You may want to distinguish between cases where short-selling is allowed and where it is not.

Hint: Use the expression for portfolio variance

Exercise 2: Opportunity sets

Again, think of the investor and the portfolio from exercise 1. Assume from now on that the assets are imperfectly correlated, $-1 < \rho_{12} < 1$.

(a) The investor's opportunity set is given by all possible combinations of the two assets (i.e. the portfolios possible when we vary a). Sketch the opportunity set in a σ, μ diagram if $a_{min} \in [0, 1]$ and if a_{min} is outside this interval. What is the *efficient frontier* of the opportunity sets?

Hint: you can draw a line at the level of μ that corresponds to the choice of a that gives the minimal variance. The graph will be symmetric around this line. What happens to the variance as you move away from this line?

(b) What is the opportunity set if short-selling is not allowed? How about when it is allowed?

(c) The opportunity set you drew above has the the same general shape even if there are n different assets, as long as there are no restrictions to portfolio creation. Then it is often called the frontier portfolio set. Why can we be sure that the graph of $\sigma(\mu)$ (the transpose of the frontier portfolio set) is convex?

Hint: Assume the opposite, that the frontier portfolio set is non-convex over some segment (has an indent somewhere). What would be the consequence for optimal portfolio choice?

Assume that the investor cares only about mean and variance, and that he or she wants to maximize the mean and minimize the variance. Also, assume that he or she finds his or her optimal portfolio (pick a point in the diagram that would be a plausible optimum). Denote the mean and variance of this optimal portfolio by $\mu(a^*)$ and $\sigma^2(a^*)$. You can also assume that short-selling is allowed.

Now let the investor gain access to a risk-free asset with return r_f.

(d) Use the efficient frontier you drew in (a) and draw the new efficient frontier when you include the possibility of investing in a risk-free asset. Show that the investor can only be made better off (or no change) by including the risk-free asset in the portfolio.

(e) Why do we never consider situations where there are more than one risk-free asset?