Problem set 4: The capital asset pricing model

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Exercise 1: Efficient portfolios with two different interest rates

Draw the frontier portfolio set when there is n different assets in a sigma-mu diagram. Assume that instead of a risk-free asset that can be freely bought or shorted, there is a bank that offers risk-free borrowing at rate r_b and risk-free saving at rate r_s. Draw the capital market line when $r_b > r_s$. How does this change the choices of investors relative to the case when there is only a single risk-free asset (you can assume that $r_s = r_f < r_b$)?

Exercise 2: The Capital Asset Pricing Model

(a) Show that the quadratic utility function $U(w) = cw^2 + bw + a$ implies mean-variance preferences when there is uncertainty over future outcomes. For what values of w does this function properly define an agent's preferences?

(b) Derive the slope of the indifference curves in a (σ, μ) diagram. Draw the indifference curves in a diagram together with the frontier portfolio set and the capital market line when there is access to a risk-free asset with rate of return r_f. Show how different degrees of risk aversion will imply different locations of the indifference curves.

(c) Assume that instead of a risk-free asset, there is a bank that offers risk-free borrowing at rate r_b and risk-free saving at rate r_s. Draw the capital market line when $r_b > r_s$. How does this change the choices of investors relative to the previous case (you can assume that $r_s = r_f < r_b$)?

(d) Derive the CAPM formula or the CAPM-β by following these steps:

1. Create a new portfolio consisting of the market portfolio and some asset j (this asset will also be part of the market portfolio). Give the market portfolio weight α and asset j weight $1-\alpha$. Find the mean and variance of this new portfolio.

2. Find the partial derivative of the mean of the portfolio with respect to α.

3. Find the partial derivative of the standard deviation of the portfolio with respect to α and evaluate it at $\alpha = 1$.

4. Use the fact that

$$\frac{\partial \mu}{\partial \sigma}\frac{\partial \sigma}{\partial \alpha} = \frac{\partial \mu}{\partial \alpha} \Leftrightarrow \frac{\partial \mu}{\partial \sigma} = \frac{\frac{\partial \mu}{\partial \alpha}}{\frac{\partial \sigma}{\partial \alpha}}$$

   Plug in the partial derivatives you found above to find $\frac{\partial \mu_p}{\partial \sigma_p}$ evaluated at $\alpha = 1$.

5. Find the slope of the capital market line.

6. Use the fact that the slope of the capital market line must be equal to $\frac{\partial \mu_p}{\partial \sigma_p}$ when $\alpha = 1$ to derive the CAPM equation

$$\mu_j = r_f + (\mu_M - r_f)\frac{\sigma_{jM}}{\sigma_M^2}$$

7. Define $\beta_j \equiv \frac{\sigma_{jM}}{\sigma_M^2}$. Rewrite the CAPM equation in terms of excess return.

You can look at appendix 8.1 in Danthine and Donaldson, but there are some typing mistakes in there, so be aware.

(e) Interpret the formula. What can we use it for?