ECON4510 – Finance Theory
Lecture 12

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Valuation of options before expiration

- Need to distinguish between American and European options.
- Consider European options with time t until expiration.
- Value now of receiving c_T at expiration? (Later also p_T.)
- Have candidate model already: Use CAPM?
- Problematic: Non-linear functions of S_T.
- Difficult to calculate $E(c_T)$ and $\text{cov}(c_T, r_M)$.
- Instead: Theory especially developed for options.
- (But turns out to have other applications as well.)
- "Valuation of derivative assets."
- Value of one asset as function of value of another.
- Will find $c(S, \ldots)$ and $p(S, \ldots)$.
- Other variables (—hopefully observable—) as arguments besides S.

Net value diagrams (Hull 9th, figs. 10.1–10.4, 12.1–12.12)

(8th ed. figs. 9.1–9.4, 11.1–11.12, 7th ed. 8.1–8.4, 10.1–10.12)

- Value at expiration *minus* purchase cost.
- S_T on horizontal axsis. (Some examples, blackboard)
- Fig. 10.1: $c_T - c$, buying a call option.
- Resembles gross value, c_T, diagram.
- But removed vertically by subtracting c, today's price.
- These diagrams only approximately true:
  - No present-value correction for time lag $-c$ to c_T.
  - There isn't one exact relationship between c and c_T.
    - That exact relationship depends on other variables.
- Fig. 10.3: $c - c_T$, selling a call option.
- Observe: Selling and buying cancel out for each S_T.
- Options redistribute risks (only). Zero-sum.

Net value diagrams, contd.

- Similar to fig. 12.10: Buy share, buy two put options with $S = K$.
- (Hull uses another combination to get the same effect in fig. 12.10.)
- Diagrams show $S_T - S$, $2p_T - 2p$, $S_T - S + 2p_T - 2p$.
- Good idea if you believe S_T will be different from S (and K), but you do not know direction.
- Fig. 12.1(c): Buy put option with $K = S$, plus one share.
- $p_T - p + S_T - S$.
- Resembles value of call option.
- Will soon show exact relationship to call option.

Determinants of option value (informally)

Six candidates for explanatory variables for c and p:

- S, today's share price. Higher S means market expects higher S_T, implies higher c (because higher c_T), lower p (lower p_T).
- K, the striking price. Higher K means lower c (because lower c_T), higher p (higher p_T).
- Uncertainty. Higher uncertainty implies both higher c *and* higher p, because option owner gains from extreme outcomes in one direction, while being protected in opposite direction. (Remark: This is total risk in S_T, not β from CAPM.)

Determinants of option value (informally), contd.

Six candidates for explanatory variables for c and p, contd.:

- Interest rate. Higher interest rate implies present value of K is reduced, increasing c, decreasing p.
- Time until expiration. Two effects (for a fixed uncertainty per unit of time): Longer time implies increased uncertainty about S_T, and lower present value of K. Both give higher c, while effects on p go in opposite directions.
- Dividends. If share pays dividends before expiration, this reduces expected S_T (for a given S, since S is claim to both dividend and S_T). Option only linked to S_T, thus lower c, higher p.

Later: Precise formula for $c(S, K, \sigma, r, t)$ when $D = 0$.

Missing from the list: $E(S_T)$. Main achievement!

Put-call parity

Exact relationship between call and put values.

- Assume underlying share with certainty pays no dividends between now and expiration date of options.
- Let $T =$ time until expiration date.
- Consider European options with same K, T.
- Consider following set of four transactions:

		At expiration	
	Now	If $S_T \leq K$	If $S_T > K$
Sell call option	c	0	$K - S_T$
Buy put option	$-p$	$K - S_T$	0
Buy share	$-S$	S_T	S_T
Borrow (risk free)	Ke^{-rT}	$-K$	$-K$
Total	$c - p - S + Ke^{-rT}$	0	0

Put-call parity, contd.

Must have $c = p + S - Ke^{-rT}$, if not, riskless arbitrage.

- To exploit arbitrage if, e.g., $c > p + S - Ke^{-rT}$:
- "Buy cheaper, sell more expensive."
- Sell (i.e., write) call option.
- Buy put option and share.
- Borrow Ke^{-rT}.
- Receive $c - p - S + Ke^{-rT} > 0$ now.
- At expiration: Net outlay zero whatever S_T is.

Put-call parity allows us to concentrate on (e.g.) calls.

Allow for uncertain dividends

- Share may pay dividends before expiration of option.
- These drain share value, do not accrue to call option.
- In Norway dividends paid once a year, in U.S., typically 4 times.
- Only short periods without dividends.
- Theoretically easily handled if dividends are known.
- But in practice: Not known with certainty.
- For short periods: $S \approx E(D + S_T)$.
- For given S, a higher D means lower S_T, lower c, higher p.
- Intuitive: High D means less left in corporation, thus option to *buy* share at K is less valuable.
- Intuitive: High D means less left in corporation, thus option to *sell* share at K is more valuable.

Allow for uncertain dividends, contd.

- Absence-of-arbitrage proofs rely on short sales.
- Short sale of shares: Must compensate for dividends.
- Short sale starts with borrowing share. Must compensate the lender of the share for the dividends missing. (Cannot just hand back share later, neglecting dividends in meantime.)
- When a-o-arbitrage proof involves shares: Could in some cases assume $D = 0$ with full certainty.
- If not $D = 0$ with certainty, conclude with inequalities instead of equalities.

More inequality results on option values

Absence-of-arbitrage proofs for American calls:

1. $C \geq 0$: If not, buy option, keep until expiration. Get something positive now, certainly nothing negative later.
2. $C \leq S$: If not, buy share, sell (i.e., write) call, receive $C - S > 0$. Get $K > 0$ if option is exercised, get S if not.
3. $C \geq S - K$: If not, buy option, exercise immediately.
4. When (for sure) no dividends: $C \geq S - Ke^{-rT}$: If not, do the following:

			Expiration	
	Now	Div. date	If $S_T \leq K$	If $S_T > K$
Sell share	S	0	$-S_T$	$-S_T$
Buy call	$-C$	0	0	$S_T - K$
Lend	$-Ke^{-rT}$	0	K	K
	> 0	0	≥ 0	0

A riskless arbitrage.

More inequality results on option values

Important implication: *American call option on shares which certainly will not pay dividends before option's expiration, should not be exercised before expiration*, since

$$C \geq S - Ke^{-rT} > S - K.$$

Worth more "alive than dead." When no dividends: *Value of American call equal to value of European*, since it is not rational to exercise these options early.

Summing up some results

Both American and European call options on shares which for sure pay no dividends:

$$C \geq S - Ke^{-rT} > S - K.$$

American call options on shares which may pay dividends:

$$C \geq S - K.$$

American calls when dividends possible: More

- For each dividend payment: Two dates.
  - One date for announcement, after which D known.
  - One *ex-dividend* date, after which share does not give the right to that dividend payment.
- Our interest is in ex-dividend dates, not announcement.
- Owners of shares on morning of ex-div. date receive D.
- Assume there is a given number of ex-div. dates.
- Say, 2 ex-div. dates, t_{d1}, t_{d2}, before option's expiration, T.
- Can show: $C > S - K$ except just before t_{d1}, t_{d2}, T.

American calls when dividends possible: More

- Assume contrary, $C \leq S - K$. Then riskless arbitrage:
- Buy call, exercise just before:

	Now	Just before next t_{di} or T
Buy call	$-C$	$S - K$
Sell share	S	$-S$
Lend	$-K$	$Ke^{r\Delta t}$
	≥ 0	$K(e^{r\Delta t} - 1)$

- Riskless arbitrage, except if $\Delta t \approx 0$, just before.

Implication: *When possible ex-dividend dates are known, American call options should never be exercised except perhaps just before one of these, or at expiration.*

Trading strategies with options, Hull 9th ed., ch. 12

(8th ed., ch. 11, 7th ed., ch. 10)

- Consider profits as functions of S_T.
- Can obtain different patterns by combining different options.
- Example: bear spread, Hull fig. 12.5
- Strategies in ch. 12 sorted like this:
  - Sect. 12.1: One option, one share.
  - Sect. 12.2: 2 or 3 calls, or 2 or 3 puts, different K values.
  - End of 12.3, pp. 265–266:[1] Different expiration dates.
  - Sect. 12.3: "Combinations", involving both puts and calls.
- Among these types of strategies, those with different expiration dates cannot be described by same method as others.

[1] 8th ed., pp. 244–245, 7th ed., pp. 227–229

Trading strategies with options, Hull 9th ed., ch. 12

(8th ed., ch. 11, 7th ed., ch. 10)

- The first, second, and fourth type:
  - Use diagram for values at expiration for each security involved.
  - Payoff at expiration is found by adding and subtracting these values.
  - Net profit is found by subtracting initial outlay from payoff.
  - Initial outlay could be negative (if, e.g., short sale of share).
  - Remember: No exact relationship between payoff and initial outlay is used in these diagrams — will depend upon, e.g., time until expiration, volatility, interest rate.
- For the third type: "Profit diagrams for calendar spreads are usually produced so that they show the profit when the short-maturity option expires on the assumption that the long-maturity option is closed out at that time" (Hull 9th ed., p. 265).[2]

[2] 8th ed., p. 244, 7th ed., p. 228

Developing an exact option pricing formula

- Exact formula based on observables very useful.
- Most used: Black and Scholes formula.
- Fischer Black and Myron Scholes, 1973.
- Their original derivation used difficult math.
- Continuous-time stochastic processes.
- First here: (Pedagogical tool:) Discrete time.
- Assume trade takes place, e.g., once per week.
- Option pricing formula in discrete time model.
- Then let time interval length decrease.
- Limit as interval length goes to zero.
- Option pricing formula in continuous time.

Common assumptions

1. No riskless arbitrage exists.
2. Short sales are allowed.
3. No taxes or transaction costs.
4. Exists a constant risk free interest rate, r.
5. Trade takes place at each available point in time. (Two different interpretations: Once per period, or continuously.)
6. S_{t+s}/S_t is stochastically independent of S_t and history before t.

Separate assumptions (discrete = d, continuous = c)

7d. S_{t+1}/S_t has two possible outcomes, u and d. Everyone agrees on these.

8d. $\Pr(S_{t+1}/S_t = u) = p^*$ for all t.

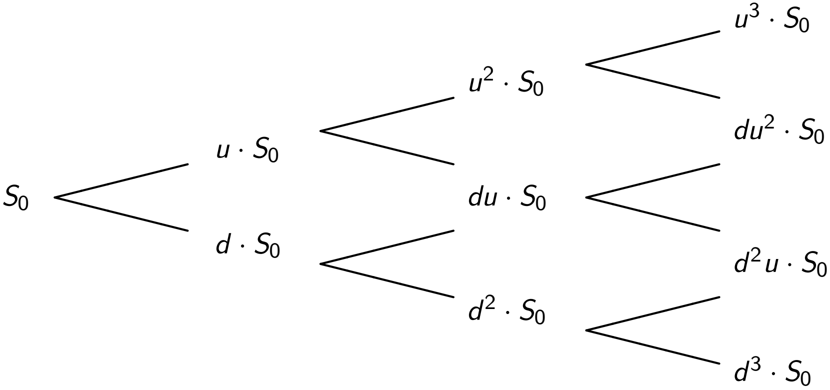
9d. S_{t+s} has a binomial distribution.

7c. Any sample path $\{S_t\}_{t=0}^{T}$ is continuous.

8c. $\text{var}[\ln(S_{t+s}/S_t)] = \sigma^2 s$. Everyone agrees on this.

9c. S_{t+s} has a lognormal distribution.

Discrete time binomial share price process



Discrete time binomial share price process

Define $X_n = S_{t+n}/S_t$. (These are stochastic variables as viewed from time t. Their distributions do not depend on t.)

$$\Pr(X_1 = u) = p^*.$$

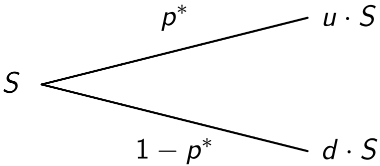
For this course we will not go into detail on the following:

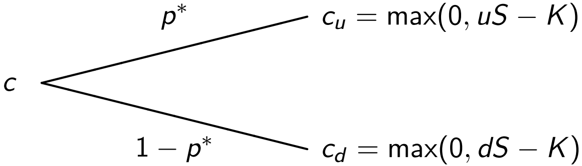
$$\Pr(X_n = u^j d^{n-j}) = \frac{n!}{j!(n-j)!} p^{*j} (1 - p^*)^{n-j},$$

the binomial probability for exactly j outcomes of one type (here u) with probability p^*, in n independent draws. ($j \leq n$.)

$$\Pr(X_n \geq u^a d^{n-a}) = \sum_{j=a}^{n} \frac{n!}{j!(n-j)!} p^{*j} (1 - p^*)^{n-j}$$

Corresponding trees for share and option





Corresponding trees for share and option

- Value of call option with expiration one period ahead?
- "Corresponding trees" mean that option value has upper outcome if and only if share value has upper outcome.
- For any K, know the two possible outcomes for c.
- I.e., for a particular option, c_u, c_d known.
- If $K \leq dS$, then $c_d = dS - K, c_u = uS - K$.
- If $dS < K \leq uS$, then $c_d = 0, c_u = uS - K$.
- If $uS < K$, then $c_d = 0, c_u = 0$.
- This third kind of option is obviously worthless.