Stylized example of project valuation

- Suppose project produces two commodities at $t = 1$.
- One variable input is needed at $t = 1$.
- Uncertain prices of input and of both commodities.
- Uncertain quantities of input and of both commodities.
- Net cash flow, $t = 1$: $\tilde{Y} = \tilde{P}_i\tilde{X}_i + \tilde{P}_j\tilde{X}_j - \tilde{P}_k\tilde{X}_k$.
- For instance, i is milk, j is beef, k is labor.
- (Warning: Many farms owned by non-diversified farmers. Then CAPM does not apply.)
- CAPM: $V(\tilde{Y}) = V(\tilde{P}_i\tilde{X}_i) + V(\tilde{P}_j\tilde{X}_j) - V(\tilde{P}_k\tilde{X}_k)$.
- Four points to be made about this:
  - Flexibility or not?
  - How to value a product of stochastic variables?
  - How to interpret valuation for negative term?
  - How to interpret valuation of, e.g., beef today?

Example, $\tilde{P}_i\tilde{X}_i + \tilde{P}_j\tilde{X}_j - \tilde{P}_k\tilde{X}_k$, contd.

Flexibility

- If any outlay at $t = 0$, those can not be cancelled later.
- What if $\tilde{Y} < 0$?
- Reasonable to assume: Each $\tilde{P}_h$ and $\tilde{X}_h$ always > 0.
- Then: $\tilde{Y} < 0$ happens when $\tilde{P}_k\tilde{X}_k$ is large.
- May be able to cancel project at $t = 1$ if $\tilde{Y} < 0$.
- If such flexibility, need option valuation methods.
- Then: Value at $t = 1$ will be 0, not $\tilde{Y}$, when $\tilde{Y} < 0$.
- Assume now: No flexibility. Committed to pay $\tilde{P}_k\tilde{X}_k$.
- For some projects, flexibility is realistic. For others, not.
- Partial flexibility may also be realistic.

Valuation of product of stochastic variables

(This will be done in the seminars next week.)

Example, $\tilde{P}_i\tilde{X}_i + \tilde{P}_j\tilde{X}_j - \tilde{P}_k\tilde{X}_k$, contd.

Valuation of negative term

$$V(-\tilde{P}_k\tilde{X}_k) = -E(\tilde{X}_k) \cdot \frac{1}{1+r_f}\left[E(\tilde{P}_k) - \lambda \operatorname{cov}(\tilde{P}_k, \tilde{r}_M)\right].$$

- If the covariance increases, then value *increases.*
- High covariance between input price and $\tilde{r}_M$ is good.
- Reason: Project owners are committed to the expense.
- Prefer expense is high when they are otherwise wealthier.
- Prefer expense is low when they are otherwise poorer.

Valuation of claim to commodity at $t = 1$

- Might perhaps calculate $V(\tilde{P}_j)$ from time series estimates of $E(\tilde{P}_j)$ and $\operatorname{cov}(\tilde{P}_j, \tilde{r}_M)$.
- "Value today of receiving one unit of beef next period."
- In general *not* equal to price of beef today.
- Would have equality if beef were investment object, like gold.
- Instead $V(\tilde{P}_j)$ is present value of *forward price* of beef.
- Usually lower than price of beef today.

CAPM: Some remarks on realism and testing

- CAPM equation may perhaps be tested on time-series data.
- More about this later in course: Roll's paper.
- Need r_f, need $\tilde{r}_M$, need stability.

Existence of risk free rate

- Interest rates on government bonds are nominally risk free.
- With inflation: Real interest rates are uncertain.
- Real rates of return are what agents really care about.
- Some countries: Indexed bonds, risk free real rates.
- Alternative model: No risk free rate. D&D sect. 6.3–6.6.
- Without r_f, still CAPM equation with testable implications.

CAPM: Some remarks on realism and testing, contd.

Observability of market portfolio

- By definition, M portfolio contains all risky assets.
- In real world, not all risky assets are traded.
- Problems of asymmetric information prevent some trading.
- E.g., many people own their homes.
- In particular: Human capital. Slavery forbidden.
- Implication: People not as well diversified as in model.
- People's risky portfolios (in extended sense) differ.
- Big problem, no good solution.

Stability of expectations, variances, covariances

- CAPM says nothing testable about single outcome.
- Need repeated outcomes, i.e., time series.
- Outcomes must be from same probability distribution.
- Requires stability over time.
- A problem, perhaps not too bad.

CAPM: Some remarks on realism and testing, contd.

- Empirical line often has too high intercept, too low slope.

- Can find other significant variables:
  - Asset-specific variables in cross-section.
  - Economy-wide variables in time series.

  If these determined at $t = 0$: Conditional CAPM.

A closer look at the CAPM

This and the next lecture:

- CAPM as equilibrium model? Mossin (1966).
- CAPM without risk free rate. D&D sect. 6.3–6.6.
- Testing the CAPM? Roll (1977) main text.

The need for an equilibrium model

- What will be effect of merging two firms?
- What will be effect of a higher interest rate?
- Could interest rate exceed $E(\tilde{r}_{mvpf})$ (min-variance-pf)?
- What will be effect of taxation?

Need equilibrium model to answer this. Partial equilibrium: Consider stock market only.

Typical competitive partial equilibrium model:

- Specify demand side: Who? Preferences?
- Leads to demand function.
- Specify supply side: Who? Preferences?
- Leads to supply function.
- Each agent views prices as exogenous.
- Supply = demand gives equilibrium, determines prices.

Repeating assumptions so far:

- Two points in time, beginning and end of period, $t = 0, 1$.
- Competitive markets. No taxes or transaction costs.
- All assets perfectly divisible.
- Agent i has exogenously given wealth W_0^i at $t = 0$.
- Wealth at $t = 1$, $\tilde{W}^i$, is value of portfolio composed at $t = 0$.
- Agent i risk averse, cares only about mean and var. of $\tilde{W}^i$.
- Portfolio composed of one risk free and many risky assets.
- Short sales are allowed.
- Agents view r_f as exogenous.
- Agents view probability distn. of risky $\tilde{r}_j$ as exogenous.
- All believe in same probability distributions.

Main results:

- CAPM equation, $\tilde{r}_j = r_f + \beta_j[E(\tilde{r}_m) - r_f]$.
- Everyone compose risky part of portfolio in same way.

Partial equilibrium model of stock market

- Main contribution of Mossin's paper.
- But have seen: Some important results without this.

Maintain all previous assumptions. Add these:

- The number of agents is m, $i = 1, \ldots, m$.
- The number of different assets is n, $j = 1, \ldots, n$.
- Before trading at $t = 0$, all assets owned by the agents: $\bar{X}_j^i$.
- After trading at $t = 0$, all assets owned by the agents: X_j^i.
- Agents own nothing else, receive no other income.
- Asset values at $t = 1$, $\tilde{p}_{j1}$, exogenous prob. distribution.
- One of these is risk free, in Mossin this happens for $j = n$.
- Choose units (for risk free bonds) so that $\tilde{p}_{n1} = p_{n1} = 1$.
- Asset values at $t = 0$, p_{j0}, endogenous for $j = 1, \ldots, n$.
- But each agent views the p_{j0}'s as exogenous.
- Thus each agent views probability distribution of $\tilde{r}_j = \tilde{p}_{j1}/p_{j0} - 1$ as exogenous.
- W_0^i consists of asset holdings, $W_0^i = \Sigma_{j=1}^n p_{j0} \bar{X}_j^i$.
- Thus each agent views own wealth, W_0^i, as exogenous.

Interpretation of model setup

- Pure exchange model. No production. No money.
- Utility attached to asset holdings.
- Market at $t = 0$ allows for reallocation of these.
- Pareto improvement: Agents trade only what they want.
- At $t = 1$ no trade, only payout of firms' realized values.

Mossin's results

- No attention to existence and uniqueness of equilibrium.
- Walras's law: Only relative prices determined.
- Choose $p_{n0} = q,\ 1 + r_f = 1/q$. Then other prices determined.
- Mossin's eq. (14): Portfolio of risky assets same for all.
- Existence of linear (σ, μ) opportunity set, CML.
- Implies: MRS between σ and μ same for all.
- No individual choice whether to locate at CML or off CML: If choose off CML, model does not hold, and CML does not exist!

Notation

Mossin, Roll and D&D use different notation:

Mossin's	D&D's and my notation	Roll's not.	Explanation
x_j^i	X_j^i		i's holding of asset j after trade
$\bar{x}_j^i$	$\bar{X}_j^i$		i's holding of asset j before trade
μ_j	$E(\tilde{p}_{j1})$		Expected price of asset j at $t = 1$
σ_j^2	$\text{var}(\tilde{p}_{j1})$		Variance of same
$1/q$	$1 + r_f$	$1 + r_F$	Risk free interest rate plus one
p_j	p_{j0}		Price of asset j at $t = 0$
w^i	W_0^i		i's wealth at $t = 0$
y_1^i	$E(\tilde{W}^i)$		i's expected wealth at $t = 1$
y_2^i	$\text{var}(\tilde{W}^i)$		Variance of same
m_j	$E(\tilde{r}_j) - r_f$	$r_j - r_F$	Asset j's expected excess return
V_j	$\text{var}(\sum_i X_j^i \tilde{p}_{j1})$		Variance of all shares of asset j
R_j	$\sum_i X_j^i p_{j0}$		Total value of asset j at $t = 0$
$\frac{y_1^i}{w^i}$	μ_p	r_p	Expected rate of return on portfolio
$\frac{\sqrt{y_2^i}}{w^i}$	σ_p	σ_p	Std. dev. of rate of return on pf.
	w_j	x_{jp}	Value weight of asset j in pf. p
	$E(\tilde{r}_z)$	r_z	Exp. rate of return on zero-β pf.

Equilibrium response to increased risk free rate?

- Previous results:

$$p_{j0} = \frac{1}{1+r_f}\left[E(\tilde{p}_{j1}) - \lambda \operatorname{cov}(\tilde{p}_{j1}, \tilde{r}_M)\right], \text{ with } \lambda = \frac{E(\tilde{r}_M) - r_f}{\operatorname{var}(\tilde{r}_M)},$$

$$E(\tilde{r}_j) = r_f + \frac{\operatorname{cov}(\tilde{r}_j, \tilde{r}_M)}{\operatorname{var}(\tilde{r}_M)}\left[E(\tilde{r}_M) - r_f\right].$$

- None of these have only exogenous variables on right-hand side.
- In both, $\tilde{r}_M$ on right-hand side is endogenous.
- Consider hyperbola and tangency in σ, μ diagram:
  - If r_f is increased, tangency point seems to move up and right.
  - Increase in $E(\tilde{r}_M)$ seems to be less than increase in r_f, and $\operatorname{var}(\tilde{r}_M)$ is increased, so $\Leftrightarrow$ increased $E(\tilde{r}_j)$?
  - But this relies on keeping hyperbola fixed.
  - CAPM equation shows that $E(\tilde{r}_j)$ is likely to change.
  - True for all risky assets, thus entire hyperbola changes.
- To detect effect of Δr_f, need only exog. variables on RHS.
- Not part of this course.

What happens if two firms merge?

- From our previous result,

  $$p_{j0} = \frac{1}{1+r_f}\left[E(\tilde{p}_{j1}) - \lambda \operatorname{cov}(\tilde{p}_{j1}, \tilde{r}_M)\right], \text{ with } \lambda = \frac{E(\tilde{r}_M) - r_f}{\operatorname{var}(\tilde{r}_M)},$$

  have shown value of merged firm equals sum of previous two values.

- Previously this was an approximate result.
- Relied on "smallness": Assumed $\tilde{r}_M$ unchanged.
- Mossin asks same question on pp. 779–781.
- More satisfactory analysis: Does not rely on $\tilde{r}_M$ unchanged.
- Get exact result: The same.
- Shows explicitly that $\tilde{r}_M$ unchanged.