Stochastic dominance

- Two criteria for making decisions without knowing shape of $U()$.
- May be important for delegation, for research, for prediction.
- Work only in a limited number of comparisons. For other comparisons, these decision criteria are inconclusive.
- Useful for narrowing down choices by excluding dominated alternatives.

First-order stochastic dominance

A random variable $\tilde{X}_A$ *first-order stochastically dominates* another random variable $\tilde{X}_B$ if every vN-M expected utility maximizer prefers $\tilde{X}_A$ to $\tilde{X}_B$.

Second-order stochastic dominance

A random variable $\tilde{X}_A$ *second-order stochastically dominates* another random variable $\tilde{X}_B$ if every *risk-averse* vN-M expected utility maximizer prefers $\tilde{X}_A$ to $\tilde{X}_B$.

First-order stochastic dominance, FSD

(Let the cumulative distribution functions be $F_A(x) \equiv \Pr(\tilde{X}_A \leq x)$ and $F_B(x) \equiv \Pr(\tilde{X}_B \leq x)$.)

Possible to show that "$\tilde{X}_A \succ \tilde{X}_B$ by all" is equivalent to the following, which is one possible definition of first-order s.d.:

$$F_A(w) \leq F_B(w) \text{ for all } w,$$

and

$$F_A(w_i) < F_B(w_i) \text{ for some } w_i.$$

For any level of wealth w, the probability that $\tilde{X}_A$ ends up below that level is less than the probability that $\tilde{X}_B$ ends up below it.

Second-order stochastic dominance, SSD

Possible to show that "$\tilde{X}_A \succ \tilde{X}_B$ by all risk averters" is equivalent to the following, which is one possible definition of second-order s.d.:

$$\int_{-\infty}^{w_i} F_A(w)dw \leq \int_{-\infty}^{w_i} F_B(w)dw \text{ for all } w_i,$$

and

$$F_A(w_i) \neq F_B(w_i) \text{ for some } w_i.$$

One distribution is more dispersed ("more uncertain") than the other. If we restrict attention to variables $\tilde{X}_A$ and $\tilde{X}_B$ with the same expected value, Theorem 3.4 in D&D states that SSD is equivalent to: $\tilde{X}_B$ can be written as $\tilde{X}_A + \tilde{z}$, where the difference $\tilde{z}$ is some random noise.

Risk aversion and simple portfolio problem

(Chapter 4 in Danthine and Donaldson.)

Simple portfolio problem, one risky, one risk free asset. Total investment is Y_0, a part of this, a, is invested in risky asset with rate of return $\tilde{r}$, while $Y_0 - a$ is invested at risk free rate r_f. Expected utility becomes a function of a, which the investor wants to maximize by choosing a:

$$W(a) \equiv E\{U[\tilde{Y_1}]\} \equiv E\{U[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}, \qquad (1)$$

based on $\tilde{Y_1} = (Y_0 - a)(1 + r_f) + a(1 + \tilde{r})$.

Solution of course depends on investor's U function. Assuming $U'' < 0$ and interior solutions we can show:

- Optimal a strictly positive if and only if $E(\tilde{r}) > r_f$.
- When the optimal a is strictly positive:
  - Optimal a independent of Y_0 for CARA, increasing in Y_0 for DARA, decreasing in Y_0 for IARA.
  - Optimal a/Y_0 independent of Y_0 for CRRA, increasing in Y_0 for DRRA, decreasing in Y_0 for IRRA.

This gives a better understanding of what it means to have, e.g., decreasing absolute risk aversion.

First-order condition for simple portfolio problem

To find f.o.c. of maximization problem (1), need take partial derivative of expectation of something with respect to a deterministic variable. Straight forward when $\tilde{r}$ has discrete probability distribution, with π_θ the probability of outcome r_θ. Then $W(a) =$

$$E\{U[Y_0(1+r_f)+a(\tilde{r}-r_f)]\} = \sum_\theta \pi_\theta U[Y_0(1+r_f)+a(r_\theta-r_f)],$$

and the f.o.c. with respect to a is

$$\begin{aligned} W'(a) &= \sum_\theta \pi_\theta U'[Y_0(1+r_f)+a(r_\theta-r_f)](r_\theta-r_f) \\ &= E\{U'[Y_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)\} = 0. \qquad (2) \end{aligned}$$

The final equation above, (2), is also f.o.c. when distribution continuous, cf. Leibniz' formula (see Sydsæter et al): The derivative of a definite integral (with respect to some variable other than the integration variable) is equal to the definite integral of the derivative of the integrand.

Observe that in (2) there is the expectation of a product, and that the two factors $U'[Y_0(1+r_f)+a(\tilde{r}-r_f)]$ and $(\tilde{r}-r_f)$ are not stochastically independent, since they depend on the same stochastic variable $\tilde{r}$. Thus this is not equal to the product of the expectations.

Prove: Invest in risky asset if and only if $E(\tilde{r}) > r_f$

Repeat: $W(a) \equiv E\{U[Y_0(1 + r_f) + a(\tilde{r} - r_f)]\}$.

Consider $W''(a) = E\{U''[Y_0(1 + r_f) + a(\tilde{r} - r_f)](\tilde{r} - r_f)^2\}$. The function $W(a)$ will be concave since U is concave. Consider now the first derivative when $a = 0$:

$$W'(0) = E\{U'([Y_0(1+r_f)](\tilde{r}-r_f)\} = U'[Y_0(1+r_f)]E(\tilde{r}-r_f). \quad (3)$$

We find:

- If $E(\tilde{r}) > r_f$, then (3) is positive, which means that E(U)=W will be increased by increasing a from $a = 0$. The optimal a is thus strictly positive.
- If $E(\tilde{r}) < r_f$, then (3) is negative, which means that E(U)=W will be increased by decreasing a from $a = 0$. The optimal a is thus strictly negative.
- If $E(\tilde{r}) = r_f$, then (3) is zero, which means that the f.o.c. is satisfied at $a = 0$. The optimal a is zero.

Of course, $a < 0$ means short-selling the risky asset, which may or may not be possible and legal.

The connection between a, Y_0, and $R_A(Y_1)$

(Theorem 4.4 in Danthine and Donaldson)

The result to prove is that the optimal a is independent of Y_0 for CARA, increasing in Y_0 for DARA, decreasing in Y_0 for IARA (assuming all the time that optimal $a > 0$).

Total differentiation of first-order condition with respect to a and Y_0:

$$E\{U''[Y_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)^2\}da$$
$$+\,E\{U''([Y_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)(1+r_f)\}dY_0 = 0$$

gives

$$\frac{da}{dY_0} = -\frac{E\{U''[Y_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)\}(1+r_f)}{E\{U''[Y_0(1+r_f)+a(\tilde{r}-r_f)](\tilde{r}-r_f)^2\}}$$

Denominator is always negative. Considering also the minus sign in front, we see that the whole expression has the same sign as the numerator. Will show this is positive for DARA. Similar proof that it is zero for CARA and negative for IARA.

da/dY_0 under Decreasing absolute risk aversion

DARA means that $R_A(Y) \equiv -U''(Y)/U'(Y)$ is a decreasing function, i.e., $R'_A(Y) < 0$ for all positive Y.

Consider first those outcomes $r_\theta > r_f$. DARA implies $R_A(Y_0(1 + r_f) + a(r_\theta - r_f)) < R_A(Y_0(1 + r_f))$, which can be rewritten

$$U''[Y_0(1+r_f)+a(r_\theta-r_f)] > -R_A(Y_0(1+r_f))U'[Y_0(1+r_f)+a(r_\theta-r_f)].$$

Multiply by the positive $(r_\theta - r_f)$ on both sides to get

$$U''[Y_0(1 + r_f) + a(r_\theta - r_f)](r_\theta - r_f)$$
$$> -R_A(Y_0(1 + r_f))U'[Y_0(1 + r_f) + a(r_\theta - r_f)](r_\theta - r_f). \quad (4)$$

Consider next outcomes $r_\theta < r_f$. DARA implies $R_A(Y_0(1 + r_f) + a(r_\theta - r_f)) > R_A(Y_0(1 + r_f))$, rewritten

$$U''[Y_0(1+r_f)+a(r_\theta-r_f)] < -R_A(Y_0(1+r_f))U'[Y_0(1+r_f)+a(r_\theta-r_f)].$$

Multiply by the negative $(r_\theta - r_f)$ on both sides to get

$$U''[Y_0(1 + r_f) + a(r_\theta - r_f)](r_\theta - r_f)$$
$$> -R_A(Y_0(1 + r_f))U'[Y_0(1 + r_f) + a(r_\theta - r_f)](r_\theta - r_f). \quad (5)$$

Clearly, (4) and (5) are the same inequality. This therefore holds for both $r_\theta > r_f$ and $r_\theta < r_f$. Then it also holds for the expectations of the LHS and the RHS: $E\{U''[Y_0(1 + r_f) + a(\tilde{r} - r_f)](\tilde{r} - r_f)\}$

$$> -R_A(Y_0(1 + r_f))E\{U'[Y_0(1 + r_f) + a(\tilde{r} - r_f)](\tilde{r} - r_f)\}$$

which is zero by the first-order condition, q.e.d.

Risk aversion and saving

(Sect. 4.6, D&D.) How does saving depend on riskiness of return? Rate of return is $\tilde{r}$, (gross) return is $\tilde{R} \equiv 1 + \tilde{r}$. Consider choice of saving, s, when probability distribution of $\tilde{R}$ is taken as given:

$$\max_{s \in R_+} E[U(Y_0 - s) + \delta U(s\tilde{R})]$$

where Y_0 is a given wealth, δ is (time) discount factor for utility.

Rewrite,

$$\max_{s \in R_+} U(Y_0 - s) + \delta E[U(s\tilde{R})],$$

first-order condition,

$$-U'(Y_0 - s) + \delta E[U'(s\tilde{R})\tilde{R}] = 0.$$

Risk aversion and saving, contd.

- Savings decision well known topic in microec. without risk
- Typical questions: Depedence of s on Y_0 and on $E(\tilde{R})$
- Focus here: How does saving depend on riskiness of $\tilde{R}$?
- Consider mean-preserving spread: Keep $E(\tilde{R})$ fixed
- Assuming risk aversion, answer is not obvious:
  - $\tilde{R}$ more risky means saving is less attractive, $\Rightarrow$ save less
  - $\tilde{R}$ more risky means probability of low $\tilde{R}$ higher, willing to give up more of today's consumption to avoid low consumption levels next period, $\Rightarrow$ save more
- Need to look carefully at first-order condition

$$U'(Y_0 - s) = \delta E[U'(s\tilde{R})\tilde{R}]$$

- What happens to right-hand side as $\tilde{R}$ becomes more risky?
- Cannot conclude in general, but for some conditions on U
- (Jensen's inequality:) Depends on concavity of $g(R) = U'(sR)R$
- If, e.g., g is concave:
  - May compare risk with no risk: $E[g(\tilde{R})] < g[E(\tilde{R})]$
  - But also some risk with more risk, cf. Theorem 4.7 in D&D.

Risk aversion and saving, contd.

$$\max_{s \in R_+} E[U(Y_0 - s) + \delta U(s\tilde{R})]$$

(assuming, all the time here, $U' > 0$ and risk aversion, $U'' < 0$)

- When $\tilde{R}_B = \tilde{R}_A + \tilde{\varepsilon}$, $E(\tilde{R}_B) = E(\tilde{R}_A)$, will show:
  - If $R_R'(Y) \leq 0$ and $R_R(Y) > 1$, then $s_A < s_B$.
  - If $R_R'(Y) \geq 0$ and $R_R(Y) < 1$, then $s_A > s_B$.
- First condition on each line concerns IRRA vs. DRRA, but both contain CRRA.
- Second condition on each line concerns magnitude of R_R (also called RRA): Higher risk aversion implies save more when risk is high. Lower risk aversion (than $R_R = 1$) implies save less when risk is high. But none of these claims hold generally; need the respective conditions on sign of R_R'.
- Interpretation: When risk aversion is high, it is very important to avoid the bad outcomes in the future, thus more is saved when the risk is increased.
- Reminder: This does *not* mean that a highly risk averse person puts more money into any asset the more risky the asset is. In this model, the portfolio choice is assumed away. If there had been a risk free asset as well, the more risk averse would save in that asset instead.

Risk aversion and saving, contd.

Proof for the first case, $R'_R(Y) \leq 0$ and $R_R(Y) > 1$:

Use $g'(R) = U''(sR)sR + U'(sR)$ and $g''(R) = U'''(sR)s^2R + 2U''(sR)s$. For g to be convex, need $U'''(sR)sR + 2U''(sR) > 0$. To prove that this holds, use

$$R'_R(Y) = \frac{[-U'''(Y)Y - U''(Y)]U'(Y) - [-U''(Y)Y]U''(Y)}{[U'(Y)]^2},$$

which implies that $R'_R(Y)$ has the same sign as

$$-U'''(Y)Y - U''(Y) - [-U''(Y)Y]U''(Y)/U'(Y)$$

$$= -U'''(Y)Y - U''(Y)[1 + R_R(Y)].$$

When $R'_R(Y) < 0$, and $R_R(Y) > 1$, this means that

$$0 < U'''(Y)Y + U''(Y)[1 + R_R(Y)] < U'''(Y)Y + U''(Y) \cdot 2.$$

(This expression has a typo in D&D, p. 70: U'' instead of U'''.) Since this holds for all Y, in particular for $Y = sR$, we find

$$U'''(sR)sR + 2U''(sR) > 0,$$

and g is thus convex.