Compulsory term paper: Suggested answers

Problem (1)

(a)

> Consider the following situation. The assumptions underlying the standard one-period CAPM are satisfied, with one exception: One agent has been denied the possibility to borrow at the risk-free interest rate (also known as short-selling the risk-free asset).
>
> Discuss what the opportunity set of the agent looks like in this case. Show this in a suitable diagram. Then discuss whether it is possible, based on the given assumptions, to narrow down in what part of the opportunity set the agent will choose his/her investment.

The CAPM relies on the following assumptions:

1. Agents' utilities depend only on the mean and the variance of their wealths at time one. When they maximize expected utility, this may rely on an assumption of normally distributed returns or on a quadratic utility function.
2. An agent's wealth at time one is the value of a portfolio of shares and riskless bonds composed from the agent's exogenous wealth at time zero.
3. The model considers competitive equilibrium in a stock market with an exogenously given interest rate. There is a fixed number of assets, which are perfectly divisible and tradable. Trade takes place only once, at time zero.
4. Everyone believes in the same, exogenously given probability distribution for the stock values at time one. All information is simultaneously available to everyone.
5. Short sales are allowed, there are no taxes and no transaction costs.

When σ is the standard deviation of the rate of return on the agent's portfolio and μ is the expected value for the same rate of return, one can describe the agent's opportunity set in a (σ, μ) diagram. If there is no risk free asset, and no possibility to construct a risk free portfolio from the risky assets, then the opportunity set can be described by an hyperbola, giving σ as a function of μ. With more than two risky assets, the interior of the hyperbola is also part of the opportunity set. For the given preferences, the efficient part of the opportunity set (— the part which all agents can agree is better than the rest of the set —) is the upper half of the hyperbola. The reason is that all prefer the maximum μ for any given σ.

When an opportunity is introduced to freely borrow and lend at a risk free interest rate, r_f, the best combination of this risk free asset with the risky ones is a ray from the r_f point on the μ axis. The ray is tangent to the hyperbola at the hyperbola's upper half. This point is called T in the first diagram. An underlying assumption is that r_f is less than the μ value corresponding to the minimum point of the hyperbola (i.e., the rightmost point when σ is on the horizontal axis). The opposite situation is neglected here.

This ray is called the Capital Market Line (CML). It is the efficient set when there is both borrowing and lending at the rate r_f, since this is now the location of point which maximizes μ for all values of σ. Along CML, a point lower than T requires a positive investment at the rate r_f and a positive investment in the risky portfolio at T, which will be the market portfolio, since this is now the combination of risky assets demanded by all agents. A point above T requires borrowing at the rate r_f to invest more than the original wealth in the risky portfolio at T.

I now consider the restriction of the choices of one agent, as mentioned in the text. This agent is denied borrowing at the rate r_f, and the part of CML above T is not available. The risky portfolios along the hyperbola northeast of T are available however, since they do not require borrowing at the rate r_f. The efficient set for this agent is thus described by the union of (i) the CML segment between r_f and T and

(ii) the part of the hyperbola extending northeast of T, in principle to infinity. This curve is thickly drawn in the first diagram. The opportunity set consist of all portfolios and assets located below the efficient set.

(b)

> Consider the following situation. The assumptions underlying the standard one-period CAPM are satisfied, with one exception: One agent has been required (by someone) to invest at least some minimum fraction of his/her wealth in the risk free alternative. The rest may be invested in risky shares. The minimum fraction is a number $a \in (0, 1)$, so that at least a fraction a of this agent's wealth must be invested in the risk free alternative.
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> Discuss what the opportunity set of the agent looks like in this case. Show this in a suitable diagram. Then discuss whether it is possible, based on the given assumptions, to narrow down in what part of the opportunity set the agent will choose his/her investment. Discuss in particular whether the agent may benefit from composing risky assets differently from the market portfolio.

The number a is given by those who impose the restriction. If the agent wants to combine this required investment in the risk free asset with investment in the tangency portfolio at T, the result would be a point on the CML segment between r_f and T, such as B in the second diagram. All points between r_f and B are available. A value of a closer to 1 would bring B down along the CML towards the vertical axis. A value of a closer to 0 would bring B up along the CML towards T. Part (a) of the problem, considered above, is the special case of $a = 0$.

However, the agent is not required to choose T as the combination of risky assets. The previous argument for choosing T do not hold anymore. By combining the required investment in the risk free asset with some other risky portfolio, other opportunities are available. In particular, it is interesting to consider those portfolios which are located on the hyperbola northeast of T. (Since part (a) above is a special case with $a = 0$, it is clear that when a is close to zero, we should look for something close to the efficient set we found in part (a).)

These combinations trace out a new curve, thickly drawn in the second diagram. We can show that this curve is also part of an hyperbola.

Let the original hyperbola through T have the equation

$$\sigma_j = \sqrt{A\mu_j^2 + B\mu_j + C}.$$

σ_j, μ_j now refer to a point above T on the original hyperbola, e.g., point R in the second diagram. Consider investing a fraction $1 - a$ in that asset and a fraction a in the risk free asset. This gives an expected rate of return

$$\mu_p = (1-a)\mu_j + ar_f \quad \Rightarrow \quad \mu_j = \frac{\mu_p - ar_f}{1-a},$$

and a standard deviation of the rate of return

$$\sigma_p = (1-a)\sigma_j \quad \Rightarrow \quad \sigma_j = \frac{\sigma_p}{1-a}.$$

If these are substituted into the equation above, we find

$$\frac{\sigma_p^2}{(1-a)^2} = A\left(\frac{\mu_p - ar_f}{1-a}\right)^2 + B\left(\frac{\mu_p - ar_f}{1-a}\right) + C.$$

This obviously gives σ_p^2 as a second-order polynomial in μ_p, which means that σ_p as a function of μ_p is an hyperbola,

$$\sigma_p = \sqrt{A(\mu_p - ar_f)^2 + B(1-a)(\mu_p - ar_f) + C(1-a)^2}.$$

The efficient set is thickly drawn in the second diagram, and the opportunity set consists of all assets and portfolios below this.

(c)

Discuss what the role of mutual funds (*aksjefond* in Norwegian) is in the standard CAPM: What kind of portfolio(s) should they offer? Then discuss what the role of mutual funds will be in relation to groups of investors who are required to obey to the kind of restrictions described in parts (a) and (b).

In the standard CAPM all agents want the same combination of risky assets, located at the tangency point T in the diagrams. This is thus equal to the market portfolio. Strictly speaking there are no transaction costs or costs of gathering information in the model. There should thus be no role for mutual funds. Small transaction costs could explain why such funds are convenient for composing risky portfolios. In the standard model, all such funds would try to compose the market portfolio. When there are restrictions, however, like in (a) or (b), some investors demand risky portfolios located on the part of the hyperbola northeast of the tangency point. This suggests that some mutual funds will try to compose portfolios like that. The appendix in Roll (1977) shows that one can find any portfolio along the hyperbola as a combination of two such portfolios. Thus there is no reason to try to offer every possible portfolio along the curve, since it is not too costly for the investor to combine two funds.

Problem (2)

Consider a corporation (joint-stock, limited company) which is going to invest in a production process which only needs input in the form of investment now, and which produces only one period from now. The produced quantity Q can be sold at a unit price $\tilde{P}$. The company does not borrow. You may assume $\theta = 1$, cf. Lund (2002).

You are asked to characterize the marginal investment under three different tax systems. More specifically, you are asked to find formulae for two numbers which characterize the marginal investment for each tax system, $i = a, b, c$. The first number is γ_i, the before-tax ratio of the valuation of the revenue to the investment. The other number is β_i, the beta value (according to the CAPM).

The three different tax systems are all special cases of the more general system analyzed in Lund (2002), except that we now need to allow $c > 1 + r_f$, contrary to Assumption 4 in Lund (2002), which says that $c \in [0, 1]$. a and c are constants. The after-tax cash flow in period 1 is

$$\tilde{V} = \tilde{P}Q - t(\tilde{P}Q - cI) = \tilde{P}Q(1 - t) + tcI. \tag{1}$$

This is based on the assumption that a fraction c of the investment I can be deducted in the company's tax base in period 1, and that this becomes effective immediately, i.e., if there is a low outcome for $\tilde{P}$ so that $\tilde{P}Q - cI < 0$, the company has other taxable income in which the amount cI can be deducted, or the authorities pay out the tax value of the (part of the) deduction which exceeds other taxable income.

The value in period 0 of having a claim to (1) is

$$\varphi(\tilde{V}) = \varphi(\tilde{P})Q(1 - t) + \frac{tcI}{1 + r_f}. \tag{2}$$

For the marginal project this value must be equal to the after-tax financing need,

$$I(1 - ta) = \varphi(\tilde{P})Q(1 - t) + \frac{tcI}{1 + r_f}. \tag{3}$$

This is based on the assumption that a fraction a of the investment I can be deducted in the company's tax base already in period 0, and that this becomes effective immediately, i.e., the company has other taxable income in which the amount aI can be deducted, or the authorities pay out the tax value of the (part of the) deduction which exceeds other taxable income. Defining $\gamma_i = \varphi(\tilde{P})Q/I$ lets us solve for γ_i,

$$\gamma_i = \frac{1}{1 - t}\left(1 - ta - \frac{tc}{1 + r_f}\right). \tag{4}$$

The beta of equity after tax is the beta of the claim to $\tilde{V}$. The cash flow $\tilde{V}$ in (1) has two parts, one risky and one risk free. The beta of the claim is a value-weighted average of the betas of these two parts (even when one of the weights is negative),

$$\beta_i = \frac{\varphi(\tilde{P})Q(1-t)}{\varphi(\tilde{V})}\beta_P + \frac{\frac{tcI}{1+r_f}}{\varphi(\tilde{V})} \cdot 0$$
$$= \frac{\varphi(\tilde{P})Q(1-t)}{\varphi(\tilde{V})}\beta_P = \frac{\gamma_i(1-t)}{1-ta}\beta_P. \tag{5}$$

(a)

Consider first a cash flow tax at a rate t. The corporation will be refunded a fraction t of the invested amount in the same period as the investment. It must pay the same fraction of the gross revenue as a tax in the next period. What will γ_i and β_i be? Interpret the result.

In this case, $a = 1$ and $c = 0$. It results in $\gamma_a = 1$ and $\beta_a = \beta_P$.

The interpretation is that this tax works cash-flow-wise just as an ownership fraction held by the authorities. The authorities must pay a fraction t of negative cash flows, but receive a fraction t of positive cash flows. Value additivity works in a similar way as between different shareholders: The fact that the remaining shareholders have to give up a fraction t of cash flows, does not change the decisions they prefer about projects in the firm. A project which has a positive net value before this tax, will also have a positive net value after tax, and vice versa.

(c) NB: REVERSE ORDER, SINCE THIS MAKES THE INTUITION SIMPLER.

Consider finally a particular kind of corporate income tax, with a rate t. This particular kind of tax will have an "allowance for corporate equity," which means that the investment of I will allow a deduction of $I(1 + r_f)$ in the next period. If the tax base becomes negative, the negative tax is refunded by the tax authorities. What will γ_i and β_i be? Interpret the result, and try to explain the differences from cases (a) and (b).

In this case, $a = 0$ and $c = 1 + r_f$. It results in $\gamma_c = 1$ and $\beta_c = (1 - t)\beta_P$.

The interpretation of $\gamma_c = 1$ is that only the present value of the tax deduction matters to shareholders. We compare with the cash flow tax in part (a). The deduction related to I is postponed one period. As long as the shareholders are sure to get the tax deduction, it is sufficient to maintain the present value of it by letting it accumulate interest. Thus a project with positive net value after tax (a) also has positive net value after tax (c), and vice versa.

However, the risk of the period-1 net cash flow is now different from that in part (a), since it now includes the tax deduction. This additional risk free positive cash flow makes the net cash flow less risky, and thus the beta is reduced. This does not imply anything about the cut-off-point for pre-tax profitability, γ, which is unaffected by the change in the tax system from (a) to (c). We observe that the beta value is simply a characteristic of the period-1 cash flow, and it does not relate to period-0 cash flow and the profitability of the project.

(b)

Consider next a tax on gross revenue, with a rate t. Whatever the investment is, none of it may be deducted, neither in the investment period nor in the production period. What will γ_i and β_i be? Interpret the result, and try to explain the difference from cases (a).

In this case, $a = 0$ and $c = 0$. It results in $\gamma_b = 1/(1 - t)$ and $\beta_b = \beta_P$.

The interpretation of γ_b is that this tax, without any deduction for I at all, requires a much higher pre-tax profitability, γ_b, in order to be exactly marginal after tax. Fewer projects will be undertaken as compared with the situation under taxes (a) or (c).

However, the risk of the period-1 net cash flow is exactly as in part (a), so the beta is the same. The change from part (a) is that the deduction of I in the tax base in period 0 has been removed, but this has no implication for the riskiness of the net cash flow in period 1.