Instructions:

For all students: Please try to solve these exercises before the seminars. It is generally a better idea to work a little with all questions than to work a lot with only a few of them.

Starting with the second meetings, 4–5 March: For those who have volunteered to present a written suggested solution: You are asked please to make copies of your solution for all participants in your seminar group. You can borrow a copy card for this purpose in the department reception office, room 1240 ES. The intention is that all students as well as the lecturer should have the opportunity to read your answers before the seminars. This means that the copies should be available before 15:00 hrs. on the preceding Monday. There are boxes, one for each seminar group, in the shelves in the students' area on the 12th floor. You are free to come and ask me (Diderik Lund) about the problems before you produce your solution. My office hour is Wednesdays 10:30–11:30, but you may contact me and arrange some other time if needed.

An answer can be hand written or computer typed. The problem with typing is that many of you are not familiar with producing formulae and diagrams on a computer. This is the main reason why we do not encourage electronic submissions, although you will have to learn some of this for writing a master's thesis.

In the texts of the exercises, the notation is sometimes inconsistent. For instance, the risk free interest rate can be r_f in problem (1), but R_f in problem (2). In some cases there is no indication of which variables are stochastic, i.e., the risky rates of return are written as r_j, not $\tilde{r}_j$. This happens because the problems are collected from various sources, and you just have to get used to this. But within each problem, there should be consistency.

Exercises for seminars 29 – 30 January

(1)

(Part of exam question for ECON460, fall 2002, slightly rewritten.)

In this exercise you are asked to sketch two different opportunity sets in the same diagram, and then to consider what can be said about what choices an investor would make.

Consider a pair of risky securities, 1 and 2, with rates of return $r_j, j = 1, 2$, with the properties $\mu_1 = E(r_1) = 0.16$, $\sigma_1^2 = \text{var}(r_1) = 0.2^2 = 0.04$, $\mu_2 = E(r_2) = 0.08$, and $\sigma_2^2 = \text{var}(r_2) = 0.3^2 = 0.09$.

Two additional pieces of information are needed in order to define the opportunity set of portfolios from the two securities. Short selling may or may not be allowed (see below). With regard to the covariance, you are asked to consider two cases:

Case (i): The covariance between the rates of return is $\sigma_{12} = \text{cov}(r_1, r_2) = 0.045$.

Case (ii): The covariance between the rates of return is $\sigma_{12} = \text{cov}(r_1, r_2) = 0.015$.

1(a)

For case (i): Determine the composition of that portfolio of the two securities which has the minimum variance, and then show that this variance is $\sigma_{mv}^2 = 0.63/16 \approx 0.198^2$. Do likewise for case (ii), with the resulting $\sigma_{mv}^2 = 0.54/16 \approx 0.184^2$.

1(b)

Sketch the two hyperbolae in the same (σ, μ) diagram. Remember that each hyperbola is symmetric around a horizontal line at the μ level which gives the minimum variance (and minimum standard deviation).

1(c)

What are the two opportunity sets (for cases (i) and (ii), resp.) if short selling is allowed? If you only know that an investor is risk averse with mean-variance preferences, can you determine whether the investor would prefer to have the opportunity set given by case (i) or that given by case (ii)?

1(d)

What are the two opportunity sets if short selling is *not* allowed? If you only know that an investor is risk averse with mean-variance preferences, can you determine whether the investor would prefer to have the opportunity set given by case (i) or that given by case (ii)?

(2)

The lecture notes for 21 January describe short selling of a risky security on p. 6. Discuss whether this description is relevant (in practice, not only in theory) for two other forms of investment: (i) A risk free security. (ii) A risky real investment project.

(3)

The lecture notes for 21 January consider situations with one or more risky assets, but there is never more than one risk free asset. Explain why a situation with many different risk free assets is not an interesting situation to consider.

(4)

In the lecture notes for 21 January it was implicitly assumed that the risky assets have different expected rates of return. What can you say about the opportunity set in a situation with only two risky assets when these have the same expected rate of return?

(5)

In the lecture on 21 January it was claimed that the frontier portfolio set, solving

$$\min_{w_1,\ldots,w_n} \sigma_p \text{ given } \mu_p$$

is a hyperbola, but there was no proof of this. For the subsequent derivation of the CAPM, however, there was no use for a formula describing the graph of the frontier portfolio set.

5(a)

Explain why the derivation must be based on an assumption (or even better, a proof) that the graph of $\sigma(\mu)$ (the frontier portfolio set transposed with μ as the argument of the function) is convex.

5(b)

Explain why we can be sure that the graph of $\sigma(\mu)$ is indeed convex. (Hint: Assume the opposite, that the curve is concave for some segment, say between μ_1 and μ_2. Why is that at odds with what you know about combining two risky assets?)

(6)

In the lectures it was claimed that when there are only two risky assets, no risk free asset, and $\rho_{12} = 1$, then the opportunity set consists of a kinked line, i.e., two rays starting at a point on the μ axis, obtained for $a = a_{\min}$. Observe that $\rho_{12} = 1$ implies that $\sigma_{12} = \sigma_1\sigma_2$. Observe also that we choose to have $\sigma_1 \leq \sigma_2$, without loss of generality (since we are free to choose the numbering of the two assets). Assume now that $\sigma_1 < \sigma_2$, i.e., the rates of return of the two assets have different standard deviations. Draw a (σ, μ) diagram and mark two points for assets 1 and 2 for the case $\mu_1 < \mu_2$. Draw a straight line between them and extend to the vertical axis.

6(a)

Show that the more general formula for $a_{\min}$ is simplified to $a_{\min} = \frac{\sigma_2}{\sigma_2 - \sigma_1}$ when $\rho_{12} = 1$. Explain why $\frac{\sigma_2}{\sigma_2 - \sigma_1} > 1$.

6(b)

Show that when $\rho_{12} = 1$, then the variance of the rate of return of a portfolio of the two assets can be written as

$$\sigma_p^2 = [a\sigma_1 + (1-a)\sigma_2]^2$$

Since the standard deviation, σ_p, is the positive number that satisfies this equation, show that as long as $a < \sigma_2/(\sigma_2 - \sigma_1)$, then $\sigma_p = a\sigma_1 + (1-a)\sigma_2 = \sigma_2 - (\sigma_2 - \sigma_1)a$. Interpret this in relation to the straight line you have drawn through the points representing the assets in the diagram. (Hint: If you have found that σ_p is a linear function of a, and you know that $a = (\mu_p - \mu_2)/(\mu_1 - \mu_2)$, then σ_p is also a linear function of μ_p.)

6(c)

Suppose you let the portfolio weight on asset 1, a, take values which are $> \sigma_2/(\sigma_2 - \sigma_1)$. What is the location in the diagram of the (rates of return of the) portfolios that you obtain in that case?

Exercises for seminars 4 – 5 March

(1a)

Consider the standard version of the CAPM, illustrated in a (σ, μ) diagram. Assume that the "risk free asset" is really represented by a bank, which offers deposits and loans at the same interest rate r_f. Illustrate in the diagram what kind of preferences will induce a person to borrow from the bank, and what kind of preferences will induce deposits.

(1b)

Assume instead that the bank offers risk free deposits with an interest rate of r_d and risk free loans with an interest rate of r_ℓ, which exceeds r_d. Show that in this situation the investors will divide themselves into three different groups, depositors, borrowers, and some who prefer not to use the bank.

(2)

(a) Consider the following situation. The assumptions underlying the standard one-period CAPM are satisfied, with one exception: One agent has been denied the possibility to borrow at the risk-free interest rate (also known as short-selling the risk-free asset).

Discuss what the opportunity set of the agent looks like in this case. Show this in a suitable diagram. Then discuss whether it is possible, based on the given assumptions, to narrow down in what part of the opportunity set the agent will choose his/her investment.

(b) Consider the following situation. The assumptions underlying the standard one-period CAPM are satisfied, with one exception: One agent has been required (by someone) to invest at least some minimum fraction of his/her wealth in the risk free alternative. The rest may be invested in risky shares. The minimum fraction is a number $a \in (0, 1)$, so that at least a fraction a of this agent's wealth must be invested in the risk free alternative. This number, a, is not chosen by the agent, but given by someone else. (Examples: The Norwegian parliament requires that a minimum fraction (at least 30 percent) of the state pension fund is invested in bonds. The University of Oslo requires something similar for its endowment funds. Similar requirements are sometimes made for donations or bequests.)

Discuss what the opportunity set of the agent looks like in this case. Show this in a suitable diagram. Then discuss whether it is possible, based on the given assumptions, to narrow down in what part of the opportunity set the agent will choose his/her investment. Discuss in particular whether the agent may benefit from composing risky assets differently from the market portfolio. (Moderately difficult: Do this graphically. More difficult: Derive equations for the opportunity set.)

(c) Discuss what the role of mutual funds (*aksjefond* in Norwegian) is in the standard CAPM: What kind of portfolio(s) should they offer? Then discuss what the role of mutual funds will be in relation to groups of investors who are required to obey to the kind of restrictions described in parts (a) and (b).

Exercises for seminars 11 – 12 March

(1)

Consider a corporation (joint-stock, limited company) which is going to invest in a production process which only needs input in the form of investment now, and which produces only one period from now. The produced quantity Q can be sold at a unit price $\tilde{P}$. The company does not borrow. You may assume $\theta = 1$, cf. Lund (2002).

You are asked to characterize the marginal investment under three different tax systems. More specifically, you are asked to find formulae for two numbers which characterize the marginal investment for each tax system, $i = a, b, c$. The first number is γ_i, the before-tax ratio of the valuation of the revenue to the investment. The other number is β_i, the beta value (according to the CAPM), relative to a no-tax situation.

(a) Consider first a cash flow tax at a rate t. The corporation will be refunded a fraction t of the invested amount in the same period as the investment. It must pay the same fraction of the gross revenue as a tax in the next period. What will γ_i and β_i be? Interpret the result.

(b) Consider next a particular kind of corporate income tax, with a rate t. This particular kind of tax will have an "allowance for corporate equity," which means that the investment of I will allow a deduction of $I(1+r_f)$ in the next period. If the tax base becomes negative, the negative tax is refunded by the tax authorities. What will γ_i and β_i be? Interpret the result, and try to explain the differences from case (a).

(c) Consider finally a tax on gross revenue, with a rate t. Whatever the investment is, none of it may be deducted, neither in the investment period nor in the production period. What will γ_i and β_i be? Interpret the result, and try to explain the difference from cases (a) and (b).

(2)

Before the Statoil-Hydro merger, Norwegian newspapers reported that the shares in the two largest Norwegian oil companies show larger relative responses to changes in the oil price than shares in (almost all) other oil companies. The question is whether we can explain this based on some observable characteristics of these companies. You are free to make simplifying assumptions, such as:

- Oil companies produce only oil, and only in one future period, next year.
- At the time when a change in share prices is observed, oil companies have already made the necessary investments this period in order to produce next period.
- The two Norwegian companies produce mainly under Norwegian taxation, while the other companies produce mainly elsewhere.

Three possible explanations are suggested below. It is OK to consider one of them at a time. That is, you may analyze borrowing as if there were no taxes and taxes as if there were no borrowing. Then you may introduce operating costs, assuming that there are no taxes and no borrowing.

(Hint: This can be done by using value additivity, with no use of, e.g., the CAPM.)

(2a)

Could the higher relative response to oil prices result from Norwegian companies having borrowed more than other companies? Or perhaps less?

(2b)

Could the higher relative response to oil prices result from high tax rates in Norway? (Hint: Consider first a cash flow tax system, then a more realistic system.)

(2c)

Could the higher relative response to oil prices result from Norwegian companies having higher operating costs next period than other companies? (Operating costs are paid simultaneously with production, in contrast with investment costs, which are paid at an earlier stage.)

Exercises for seminars 1 – 2 April

(1)

Consider the following two alternative stochastic processes, suggested as descriptions of a stock price.

$$P_t = P_{t-1} \cdot u_t \quad \text{with } u_t \text{ i.i.d.} \tag{1}$$

$$P_t = P_0 \cdot e^{a \cdot t} \cdot v_t \quad \text{with } v_t \text{ i.i.d.} \tag{2}$$

You can assume that a is a positive constant, and that $E_{t-1}(v_t) = 1$. You may also, as a simplification, assume that the stock has a beta value of zero.

Discuss whether the two processes have the same expected time path (for some values of a and $E_0(u_t)$). Discuss for each process whether it could describe stock prices in an efficient stock market. If not, how could you device a trading rule to make profits based on observing outcomes? (Moderately difficult: Suggest a rule, and explain in words why it will work. More difficult: Suggest a rule, and prove with calculations that the expected logarithm of the return will represent an excess return in relation to the risk.)

(2)

In lecture no. 7, Shiller's test of market efficiency is presented as follows: If a stock price is an unbiased, optimal forecast of the expected present value of dividends, then it can be written as

$$P_t = \sum_{\tau=1}^{\infty} \frac{E_t(D_{t+\tau})}{(1+r_a)^{\tau}}, \tag{3}$$

under the assumption that the risk adjusted discount rate r_a is constant. The realized present values of dividends can be written as expected values plus an error term U_t with expectation equal to zero,

$$\sum_{\tau=1}^{\infty} \frac{D_{t+\tau}}{(1+r_a)^{\tau}} = \sum_{\tau=1}^{\infty} \frac{E_t(D_{t+\tau})}{(1+r_a)^{\tau}} + U_t.$$

Shiller observes that the optimal forecast should be uncorrelated with the error term. Thus he arrives at the hypothesis

$$\text{var}\left[\sum_{\tau=1}^{\infty} \frac{D_{t+\tau}}{(1+r_a)^{\tau}}\right] = \text{var}\left[P_t\right] + \text{var}\left[U_t\right] > \text{var}\left[P_t\right],$$

which is rejected by the data.

(a) Suppose instead that the market makes some unsystematic mistakes, so that

$$P_t = \sum_{\tau=1}^{\infty} \frac{E_t(D_{t+\tau})}{(1+r_a)^{\tau}} + V_t,$$

where V_t are i.i.d., with expectation zero. Discuss whether this could explain the variance result which Shiller finds. Discuss whether this could be used to device a trading rule in order to make profit based on observations of the outcomes.

(3)

Introduce a growth rate, g, in equation (3) above, and show that it can be reformulated into the formula on top of page 74 in Malkiel's article. (Depending on the details, you may get a factor $(1+g)$ too much.) If the required expected rate of return did *not* change as suggested by Malkiel, how much would g have to change instead in order to justify a 33 percent drop in share prices?

(4)

Suppose that you live in an economy with efficient capital markets. You have one piece of information which is not known to the market: One of your fellow students, who is very clever and (for some reason) got a master's in engineering, not in economics, started right after university to work for a company which is traded in the stock market. A year later it is announced that the company has developed a new technology which will save costs significantly. Another year later it is announced that the company has sold licenses for this new technology to all its competitors. Discuss at what points in time it is a good idea for you to buy and/or sell shares in the company.

Exercises for seminars 15 – 16 April

Consider the function $U(C) \equiv ae^{bC} + d$, where a, b, and d are constants, and $e \approx 2.78$ is the well-known constant. C is consumption.

(a)

What condition(s) must be satisfied by a, b, and d in order for $E[U(C)]$ to properly represent the preferences of a risk averse person who maximizes von Neumann-Morgenstern expected utility? What are the coefficients of absolute and relative risk aversion, $R_A(C)$ and $R_R(C)$, for this U function?

Consider an individual planning for a future period, with the given U function with that/those conditions you stated in part (a). The individual has wealth Y_0 to be divided between two financial investments, Y_f (risk free) and Y_r (risky). The future consumption is equal to the sum of the future values of these investments. Y_f will increase by the factor $R_f \equiv 1 + r_f$, where r_f is the risk free interest rate. Y_r will increase similarly by the factor $\tilde{R} \equiv 1 + \tilde{r}$, where $\tilde{r}$ is a risky rate of return (— this may in fact be a decrease for low outcomes of $\tilde{r}$). The individual regards Y_0, R_f, and the probability distribution of $\tilde{R}$ as exogenously given.

In what follows, you should discuss both the case in which short selling is allowed and the case in which it is not.

(b)

Describe the individual's maximization problem and its solution.

(c)

Discuss the statement "Optimal Y_r does not depend on the size of Y_0."

Assume in the following that $\tilde{R}$ has only two possible outcomes, R_1 and R_2, and that $R_1 > R_2$. The probability of R_1 is p.

(d)

Describe the solution to the maximization problem in this case. Show that under some conditions the optimal Y_r can be written as

$$Y_r^* = \frac{\ln \frac{p(R_1 - R_f)}{(1-p)(R_f - R_2)}}{-b(R_1 - R_2)}.$$

(e)

Find out whether Y_r^* as given by the formula above is increased or decreased by changes in p, R_f, and b. Try to give intuitive explanations for these effects.

(f)

Could Y_r^* from the formula exceed Y_0? What would the individual do then if borrowing is not allowed?

(g)

Could Y_r^* from the formula be negative? What would the individual do then if short selling is not allowed?

(h)

What will happen if $R_1 < R_f$? What will happen if $R_2 > R_f$? Can these situations occur?

(i)

What would be the solution for parts (a) – (d) if the individual is instead attracted to risk?

Exercises for seminars 29 – 30 April

(1)

(From exam spring 1999.) Discuss whether the following observations can be consistent with the CAPM, or alternatively, what could be the reason(s) to observe deviations from the model. Discuss each point separately:

(a) Many agents choose to invest only in risk free bank accounts, not in the stock market.

(b) Some agents choose a more risky portfolio of shares than other agents. They say they do this because they are willing to take more risk in order to obtain a higher return.

(2)

In the zero-beta CAPM, find the location in the (σ, μ) diagram of all assets j which have

(a) $\text{cov}(\tilde{r}_j, \tilde{r}_m) = 1$

(b) $\text{cov}(\tilde{r}_j, \tilde{r}_m) = 0$

(c) $\text{cov}(\tilde{r}_j, \tilde{r}_m) = 0.5$

(Consider each of these three cases separately.)

(3)

Consider the article by Roll (1977).

(a) What is meant by an ex post efficient portfolio? If you were able to identify such a portfolio, would that make it easier to test the CAPM?

(b) Assume that a riskless interest rate exists. If you look at weekly data for returns on assets and portfolios, you may perhaps discover that during some weeks, the realized rate of return on the market portfolio, r_m, is less than the riskless interest rate, r_f. Does this contradict the CAPM?

(4)

Consider an economy with only two individuals and one future period, in which only two possible states of the world may be realized. The two individuals both have von Neumann-Morgenstern preferences and are risk averse. They both agree that the states have probabilities π_1 and $\pi_2 = 1 - \pi_1$. Individual i will have an exogenously given income Y_{is} in state s, but may instead obtain consumption c_{is} in that state by buying and selling state-contingent claims in a market. Only the two individuals take part in the trade, but we nevertheless suppose that a competitive equilibrium is obtained. Each individual's budget for buying the claims consists solely of the equilibrium values of that individual's incomes, Y_{is}, i.e., $\hat{Y}_i$ is the budget of individual i, $\hat{Y}_i \equiv p_1 Y_{i1} + p_2 Y_{i2}$, where p_s is the price of a claim to one unit income in state s.

4(a)

Write down the optimalization problem for individual i, and find the first-order conditions for a maximum, without specifying the shape of the utility function. Why can we assume that the second-order conditions are satisfied?

4(b)

Assume both individuals have a utility function of the form $U_i(c_{is}) \equiv -e^{-\alpha_i c_{is}}$, where α_i is an individual-specific, positive constant. Show that under the assumption that there is an interior solution, the optimal c_{i1} can be written as

$$c_{i1} = \frac{\hat{Y}_i - \frac{p_2}{\alpha_i} \ln\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)}{p_1 + p_2}$$

4(c)

Define $\alpha \equiv 1/\left(1/\alpha_1 + 1/\alpha_2\right)$, and define Y_s (with only one subscript, without hat) as $Y_s \equiv Y_{1s} + Y_{2s}$. Show that in equilibrium the relative price is given as

$$p = \frac{p_1}{p_2} = \frac{\pi_1}{\pi_2} e^{\alpha(Y_2 - Y_1)}$$

4(d)

Give an economic interpretation of how the equilibrium relative price depends on π_1/π_2, on $Y_2 - Y_1$, and on α_1 and α_2.

4(e)

Assume now $Y_1 = Y_2$. What is now the formulae for the relative price and the optimal consumptions? Give an economic interpretation of this case.

4(f)

Assume now that in addition to the future period discussed above, there is a period we may call "today." Consider two investment projects which require the same outlay today. The first gives an income x in the future period, the same irrespective of which state is realized. The second gives an income z in state 1, but nothing in state 2. What do the equilibrium prices from (c) above tell us about (i) the agents' rankings of these two projects, and (ii) the agents' willingness today to pay for the projects (i.e., what are the maximum outlays today that would lead the agents to accept the projects)?