ECON4515 Finance theory 1 Diderik Lund, 4 February 2008

Reference to articles on CAPM with nonmarketable assets

There was a question during the lecture on 28 January about possible deviations from the assumptions underlying the CAPM. In my answer I mentioned one important deviation, the existence of future uncertain income from sources that are not tradeable in markets. A typical example is labor income. Claims to your own future labor income cannot be sold today, partly because slavery is illegal, partly because of severe incentive problems. Another example may be housing. Many people are poorly diversified because they own their own house. Again, incentive problems may be part of the explanation.

Some references are

- David Mayers (1972), "Nonmarketable assets and capital market equilibrium under uncertainty," in Michael C. Jensen, ed., *Studies in the Theory of Capital Markets,* New York, Praeger, pp. 223–248.
- David Mayers (1973), "Nonmarketable assets and the determination of capital asset prices in the absence of a riskless asset," *Journal of Business,* vol. 46, pp. 258–267.
- Ney O. Brito (1977), "Marketability restrictions and the valuation of capital assets under uncertainty," *Journal of Finance,* vol. 32, no. 4, pp. 1109–1123.
- Diderik Lund (1993), "Usikre investeringer under begrenset diversifisering," *Beta,* vol. 7, no. 2, pp. 14-21.

The first of these is the most easily readable, but probably not available online. However, you may use scholar.google.com to find 150 other articles which have cited it.

The fourth is in Norwegian, but a version in English is now available on the course semester web page. It contains an extension to find the willingness to pay for more of the nonmarketable asset.

Stylized example of project valuation

- Suppose project produces two commodities at $t = 1$.
- One variable input is needed at $t = 1$.
- Uncertain prices of input and of both commodities.
- Uncertain quantities of input and of both commodities.
- Net cash flow, $t = 1$: $\tilde{p}_{I1} = \tilde{P}_i\tilde{X}_i + \tilde{P}_j\tilde{X}_j - \tilde{P}_k\tilde{X}_k$.
- For instance, i is milk, j is beef, k is labor.
- (Warning: Many farms owned by poorly diversified farmers. Then standard CAPM does not apply, but see previous page.)
- CAPM: $V(\tilde{p}_{I1}) = V(\tilde{P}_i\tilde{X}_i) + V(\tilde{P}_j\tilde{X}_j) - V(\tilde{P}_k\tilde{X}_k)$.
- Four points to be made about this:
  - Flexibility or not?
  - How to value a product of stochastic variables?
  - How to interpret valuation for negative term?
  - How to interpret valuation of, e.g., beef today?

Example, $\tilde{p}_{I1} = \tilde{P}_i\tilde{X}_i + \tilde{P}_j\tilde{X}_j - \tilde{P}_k\tilde{X}_k$, contd.

Flexibility

- If any outlay at $t = 0$, those can not be cancelled later.
- What if $\tilde{p}_{I1} < 0$?
- May assume: Each $\tilde{P}_h$ and $\tilde{X}_h$ always > 0 for $h = i, j, k$.
- Then: $\tilde{p}_{I1} < 0$ happens when $\tilde{P}_k\tilde{X}_k$ is large.
- May be able to cancel project at $t = 1$ if $\tilde{p}_{I1} < 0$.
- If such flexibility, need option valuation methods.
- Then: Value at $t = 1$ will be 0, not $\tilde{p}_{I1}$, when $\tilde{p}_{I1} < 0$.
- Assume now: No flexibility. Committed to pay $\tilde{P}_k\tilde{X}_k$.
- For some projects, flexibility is realistic. For others, not.
- Perhaps partial flexibility would be most realistic.

Valuation of product of stochastic variables

Quantity uncertainty often local, technical, meteorological. May simplify valuation of $\tilde{P}\tilde{X}$ expressions if assume: Each $\tilde{X}_h$ ($h = i, j, k$) is stoch. indep. of $(\tilde{P}_h, \tilde{r}_M)$. Then: $E(\tilde{P}\tilde{X}) = E(\tilde{P})E(\tilde{X})$ and

$$\text{cov}(\tilde{P}\tilde{X}, \tilde{r}_M) = E(\tilde{P}\tilde{X}\tilde{r}_M) - E(\tilde{P}\tilde{X})E(\tilde{r}_M)$$
$$= E(\tilde{X})\left[E(\tilde{P}\tilde{r}_M) - E(\tilde{P})E(\tilde{r}_M)\right] = E(\tilde{X})\,\text{cov}(\tilde{P}, \tilde{r}_M) \Rightarrow$$

$V(\tilde{P}\tilde{X}) = E(\tilde{X})V(\tilde{P})$, quantity uncertainty irrelevant.

Example, $\tilde{P}_i\tilde{X}_i + \tilde{P}_j\tilde{X}_j - \tilde{P}_k\tilde{X}_k$, contd.

Valuation of negative term

$$V(-\tilde{P}_k\tilde{X}_k) = -E(\tilde{X}_k) \cdot \frac{1}{1+r_f}\left[E(\tilde{P}_k) - \lambda\,\mathrm{cov}(\tilde{P}_k, \tilde{r}_M)\right].$$

- If the covariance increases, then value *increases.*
- High covariance between input price and $\tilde{r}_M$ is good.
- Reason: Project owners are committed to the expense.
- Prefer expense is high when they are otherwise wealthier.
- Prefer expense is low when they are otherwise poorer.

Valuation of claim to commodity at $t = 1$

- Might perhaps calculate $V(\tilde{P}_j)$ from time series estimates of $E(\tilde{P}_j)$ and $\mathrm{cov}(\tilde{P}_j, \tilde{r}_M)$.
- "Value today of receiving one unit of beef next period."
- In general *not* equal to price of beef today.
- Would have equality if beef were investment object, like gold.
- Instead $V(\tilde{P}_j)$ is present value of *forward price* of beef.
- Usually lower than price of beef today.

CAPM: Some remarks on realism and testing

- CAPM equation may perhaps be tested on time-series data.
- More about this later in course: Roll's paper.
- Need r_f, need $\tilde{r}_M$, need stability.

Existence of risk free rate

- Interest rates on government bonds are nominally risk free.
- With inflation: Real interest rates are uncertain.
- Real rates of return are what agents really care about.
- Some countries: Indexed bonds, risk free real rates.
- Alternative model: No risk free rate. D&D sect. 6.3–6.6.
- Without r_f, still CAPM equation with testable implications.

Stability of expectations, variances, covariances

- CAPM says nothing testable about single outcome.
- Need repeated outcomes, i.e., time series.
- Outcomes must be from same probability distribution.
- Requires stability over time.
- A problem, perhaps not too bad.

CAPM: Some remarks on realism and testing, contd.

- Empirical line often has too high intercept, too low slope.

- Can find other significant variables:
  - Asset-specific variables in cross-section.
  - Economy-wide variables in time series.

  If these determined at $t = 0$: Conditional CAPM.

A closer look at the CAPM

This and the next lecture:

- CAPM as equilibrium model? Mossin (1966).
- CAPM without risk free rate. D&D sect. 6.3–6.6.
- Testing the CAPM? Roll (1977) main text.

The need for an equilibrium model

- What will be effect of merging two firms?
- What will be effect of a higher interest rate?
- Could interest rate exceed $E(\tilde{r}_{mvpf})$ (min-variance-pf)?
- What will be effect of taxation?

Need equilibrium model to answer this. Partial equilibrium: Consider stock market only.

Typical competitive partial equilibrium model:

- Specify demand side: Who? Preferences?
- Leads to demand function.
- Specify supply side: Who? Preferences?
- Leads to supply function.
- Each agent views prices as exogenous.
- Supply = demand gives equilibrium, determines prices.

Repeating assumptions so far:

- Two points in time, beginning and end of period, $t = 0, 1$.
- Competitive markets. No taxes or transaction costs.
- All assets perfectly divisible.
- Agent i has exogenously given wealth W_0^i at $t = 0$.
- Wealth at $t = 1$, $\tilde{W}^i$, is value of portfolio composed at $t = 0$.
- Agent i risk averse, cares only about mean and var. of $\tilde{W}^i$.
- Portfolio composed of one risk free and many risky assets.
- Short sales are allowed.
- Agents view r_f as exogenous.
- Agents view probability distn. of risky $\tilde{r}_j$ as exogenous.
- All believe in same probability distributions.

Main results:

- CAPM equation, $\tilde{r}_j = r_f + \beta_j[E(\tilde{r}_m) - r_f]$.
- Everyone compose risky part of portfolio in same way.

Partial equilibrium model of stock market

- Main contribution of Mossin's paper.
- But have seen: Some important results without this.

Maintain all previous assumptions. Add these:

- The number of agents is m, $i = 1, \ldots, m$.
- The number of different assets is n, $j = 1, \ldots, n$.
- Before trading at $t = 0$, all assets owned by the agents: $\bar{X}_j^i$.
- After trading at $t = 0$, all assets owned by the agents: X_j^i.
- Agents own nothing else, receive no other income.
- Asset values at $t = 1$, $\tilde{p}_{j1}$, exogenous prob. distribution.
- One of these is risk free, in Mossin this happens for $j = n$.
- Choose units (for risk free bonds) so that $\tilde{p}_{n1} = p_{n1} = 1$.
- Asset values at $t = 0$, p_{j0}, endogenous for $j = 1, \ldots, n$.
- But each agent views the p_{j0}'s as exogenous.
- Thus each agent views probability distribution of $\tilde{r}_j = \tilde{p}_{j1}/p_{j0} - 1$ as exogenous.
- W_0^i consists of asset holdings, $W_0^i = \Sigma_{j=1}^n p_{j0}\bar{X}_j^i$.
- Thus each agent views own wealth, W_0^i, as exogenous.

Interpretation of model setup

- Pure exchange model. No production. No money.
- Utility attached to asset holdings.
- Market at $t = 0$ allows for reallocation of these.
- Pareto improvement: Agents trade only what they want.
- At $t = 1$ no trade, only payout of firms' realized values.

Mossin's results

- No attention to existence and uniqueness of equilibrium.
- Walras's law: Only relative prices determined.
- Choose $p_{n0} = q,\ 1 + r_f = 1/q$. Then other prices determined.
- Mossin's eq. (14): Portfolio of risky assets same for all.
- Existence of linear (σ, μ) opportunity set, CML.
- Implies: MRS between σ and μ same for all.
- No individual choice whether to locate at CML or off CML: If choose off CML, model does not hold, and CML does not exist!

Notation

Mossin, Roll and D&D use different notation:

Mossin's	D&D's and my notation	Roll's not.	Explanation
x^i_j	X^i_j		i's holding of asset j after trade
$\bar{x}^i_j$	$\bar{X}^i_j$		i's holding of asset j before trade
μ_j	$E(\tilde{p}_{j1})$		Expected price of asset j at $t = 1$
σ^2_j	$\text{var}(\tilde{p}_{j1})$		Variance of same
$1/q$	$1 + r_f$	$1 + r_F$	Risk free interest rate plus one
p_j	p_{j0}		Price of asset j at $t = 0$
w^i	W^i_0		i's wealth at $t = 0$
y^i_1	$E(\tilde{W}^i)$		i's expected wealth at $t = 1$
y^i_2	$\text{var}(\tilde{W}^i)$		Variance of same
m_j	$E(\tilde{r}_j) - r_f$	$r_j - r_F$	Asset j's expected excess return
V_j	$\text{var}(\sum_i X^i_j \tilde{p}_{j1})$		Variance of all shares of asset j
R_j	$\sum_i X^i_j p_{j0}$		Total value of asset j at $t = 0$
$\frac{y^i_1}{w^i}$	μ_p	r_p	Expected rate of return on portfolio
$\frac{\sqrt{y^i_2}}{w^i}$	σ_p	σ_p	Std. dev. of rate of return on pf.
	w_j	x_{jp}	Value weight of asset j in pf. p
	$E(\tilde{r}_z)$	r_z	Exp. rate of return on zero-β pf.

Equilibrium response to increased risk free rate?

- Previous results:

$$p_{j0} = \frac{1}{1+r_f}\left[E(\tilde{p}_{j1}) - \lambda \operatorname{cov}(\tilde{p}_{j1}, \tilde{r}_M)\right], \text{ with } \lambda = \frac{E(\tilde{r}_M) - r_f}{\operatorname{var}(\tilde{r}_M)},$$

$$E(\tilde{r}_j) = r_f + \frac{\operatorname{cov}(\tilde{r}_j, \tilde{r}_M)}{\operatorname{var}(\tilde{r}_M)}\left[E(\tilde{r}_M) - r_f\right].$$

- None of these have only exogenous variables on right-hand side.
- In both, $\tilde{r}_M$ on right-hand side is endogenous.
- Consider hyperbola and tangency in σ, μ diagram:
  - If r_f is increased, tangency point seems to move up and right.
  - Increase in $E(\tilde{r}_M)$ seems to be less than increase in r_f, and $\operatorname{var}(\tilde{r}_M)$ is increased, so $\Leftrightarrow$ increased $E(\tilde{r}_j)$?
  - But this relies on keeping hyperbola fixed.
  - CAPM equation shows that $E(\tilde{r}_j)$ is likely to change.
  - True for all risky assets, thus entire hyperbola changes.
- To detect effect of Δr_f, need only exog. variables on RHS.
- Not part of this course.

What happens if two firms merge?

- From our previous result,

  $$p_{j0} = \frac{1}{1+r_f}\left[E(\tilde{p}_{j1}) - \lambda \operatorname{cov}(\tilde{p}_{j1}, \tilde{r}_M)\right], \text{ with } \lambda = \frac{E(\tilde{r}_M) - r_f}{\operatorname{var}(\tilde{r}_M)},$$

  have shown value of merged firm equals sum of previous two values.

- Previously this was an approximate result.
- Relied on "smallness": Assumed $\tilde{r}_M$ unchanged.
- Mossin asks same question on pp. 779–781.
- More satisfactory analysis: Does not rely on $\tilde{r}_M$ unchanged.
- Get exact result: The same.
- Shows explicitly that $\tilde{r}_M$ unchanged.