Perold: The CAPM

- Perold starts with a historical background, the development of portfolio theory and the CAPM.
- Points out that until 1950 there was no theory to describe the equilibrium determination of asset prices.
- Then started a theoretical development for about 15 years.
- After the CAPM was established in the mid-1960's, subsequent development has been related to that model: Extending it, testing it, and (see Fama and French) rejecting it.
- The central role of the CAPM during these forty years is one good reason to learn about it.
- Other reasons to learn about the CAPM:
  - Starting from a simple assumption of mean-variance preferences, the model derives a formula for the pricing of assets which can be applied by practitioners and tested by econometricians.
  - The pricing formula is linear, in a sense which is necessary for capital market equilibrium: The price of a linear combination of assets is equal to the same linear combination of the separate prices of those assets.
  - The model contains the very general lesson that when there are diversification possibilities, the required expected rate of return for each asset does not depend on the asset's own risk, but its contribution to portfolio risk, measured by a covariance.

Perold: The CAPM, continued

- Observe: The linearity must hold because the agents are free to make such linear combinations themselves. Equilibrium prices cannot be determined by variance, since the price would then increase each time two assets were combined into a portfolio so that variance was reduced.
- Observe: The linearity holds when the risk deduction in the price is determined by a covariance, but would not hold if it were determined by, e.g., a variance.
- Observe: The linearity will also be satisfied by Fama and French's three-factor model, p. 38 in their article.

Perold's derivation of the CAPM equation, pp. 9–17

- This is an alternative derivation to the one in D&D, p. 142, which was also used in the lectures (28 Jan., p. 1).
- Perold's derivation uses exactly the same assumptions and ends up with the same equation.
- Thus no need to learn Perold's method in addition to the one we used previously.
- But notice the concept Sharpe ratio, which is much used.
- Equal to the slope in the (σ, μ) diagram of the line through $(0, r_f)$ and the location (σ_j, μ_j) in the diagram of some asset or portfolio.
- In the standard CAPM, the maximal attainable Sharpe ratio is equal to the slope of the Capital Market Line.

Other topics from Perold

- If the CAPM does not describe the world correctly, it may help in showing what could be gained from behavior closer to the model's predictions.
- In particular (pp. 18–19) Perold suggests much could be gained from better international diversification.
- If there were no obstacles to international diversification, the model predicts that all investors would hold the same combination of risky assets.
- This is definitely not true for the world. Explanations could be tax discrimination or information problems.

Fama and French (2004)

- Paper contains some econometric arguments which are too advanced for this course.
- Since not all of you have masters' courses in econometrics, we cannot require that you understand and reproduce the econometric arguments.
- However, you should know in more general terms what are their viewpoints on the empirical validity of the CAPM.
- Main viewpoint is that the model has serious problems in passing empirical tests.

F&F on the Roll critique

- F&F acknowledge the Roll critique, that there are important problems with the methods used to test the CAPM.
- Although it is difficult to know which empirical data to use for the market portfolio, F&F state that they "are more pragmatic" (than Roll) (p. 41).
- My interpretation: They observe that the model is being used by many with some particular choice(s) of market portfolio(s). Thus they say it is interesting to test whether the model works with those empirical specifications that people actually use.
- Example: People may estimate the β for some industry, say production of furniture (or automobiles or cement or . . .). The estimate is based on some choice of market portfolio. Then they will use that β to determine the required expected rate of return for projects in the industry. This will be used to make investment decisions.
- F&F say (see, e.g., the second paragraph in their conclusion) that if one uses such a procedure, one may use a wrong required expected rate of return, because the model does not conform with data, using that specific market portfolio.

F&F's three-factor model

- According to the CAPM, no other variable should affect differences in expected returns than the covariance with the return on the market portfolio.
- (Assets with the same covariance will have the same $E(\tilde{r}_j)$, according to the model. Differences in other variables between these assets cannot then affect the $E(\tilde{r}_j)$.)
- In empirical tests, many researchers have tried to find if other factors can explain differences.
- Use different regression techniques, enter new explanatory variables.
- F&F (pp. 38–41) have the most influential results so far: Three factors.
- Beside the market factor, they enter two additional factor in the regression.
- One is the difference in the return on portfolios of small and big stocks.
- The other is the difference in the return on portfolios of stocks with high and low B/M ratios (see p. 10 of lecture 21 April).
- Will not go further into explanations and further testing of these.
- But problem (F&F p. 39): No theory underlying the inclusion of these variables. No idea whether they will continue to be significant.

- One possible reason why B/M matters is that there may be some irrational pricing.
- If we allow for the possibility that the market sometimes underprice or overprice an asset, then a high price today will predict lower expected rate of return over the next period, and vice versa.
- This will imply that every variable which includes the market price itself, will be significant in explaining expected rates of return.

Seminar exercise 4 for 29–30 April

- One future period with only two possible states of the world, $s = 1, 2$, with probabilities $\Pr(S = s) = \pi_s$, so that $\pi_2 = 1 - \pi_1$.
- Only two individuals, $i = 1, 2$, who maximize expected utility and are risk averse.
- Each individual, i, has an endowment of income Y_{is} in state s for $s = 1, 2$. All four $Y_{is} \geq 0$.
- Before the state of the world is known, they can trade in claims to the income in these two states. Competitive equilibrium.
- Individual i will consume c_{is} in state s. The budget constraint of individual i is

$$\hat{Y}_i \equiv p_1 Y_{i1} + p_2 Y_{i2} = p_1 c_{i1} + p_2 c_{i2},$$

  where $\hat{Y}_i$ is just a definition of the budget.
- This can be rewritten in terms of net trades:

$$p_1(Y_{i1} - c_{i1}) = p_2(c_{i2} - Y_{i2}).$$

  By selling claims to some of the income in one state it is possible to buy claims to consumption exceeding the income in the other state.
- Observe that compared to the model in p. 12 of the lecture of 28 April, there is no endowment (wealth, income) or consumption in period zero in the present model. But the market allows for the same kind of trading (in both models) between states in period one (the future time period).

4(a)

Write down the optimalization problem for individual i, and find the first-order conditions for a maximum, without specifying the shape of the utility function. Why can we assume that the second-order conditions are satisfied?

Answer

The problem is to maximize expected utility given the budget constraint,

$$\max_{c_{i1},c_{i2}} \pi_1 U_i(c_{i1}) + \pi_2 U_i(c_{i2}),$$

s.t.

$$p_1 Y_{i1} + p_2 Y_{i2} = p_1 c_{i1} + p_2 c_{i2}.$$

Substitute in for c_{i2} from the budget constraint and find

$$\max_{c_{i1}} \pi_1 U_i(c_{i1}) + \pi_2 U_i(Y_{i2} + \frac{p_1}{p_2}(Y_{i1} - c_{i1})),$$

which has f.o.c. (chain rule, dc_{i2}/dc_{i1} is $-p_1/p_2$):

$$\frac{\pi_1 U_i'(c_{i1})}{\pi_2 U_i'(c_{i2})} = \frac{p_1}{p_2}$$

(Observe that a similar f.o.c. can be derived from the f.o.c. in p. 12 of the lecture (28 Apr) since there, $u'(u_0)$ must be equal to $\pi_\theta U'(c_\theta)/q_\theta$ for both $\theta = 1$ and $\theta = 2$.)

The s.o.c. is that

$$\pi_1 U_i''(c_{i1}) + \pi_2 U_i''(c_{i2}) \left(\frac{p_1}{p_2}\right)^2 < 0$$

which is OK since $U'' < 0$.

(I am sorry for the typos in the original version of 4(b) and (c). It was corrected in a message 29 April.)

4(b)

Assume both individuals have a utility function of the form $U_i(c_{is}) \equiv -e^{-\alpha_i c_{is}}$, where α_i is an individual-specific, positive constant. Show that under the assumption that there is an interior solution, the optimal c_{i1} can be written as

$$c_{i1} = \frac{\hat{Y}_i - \frac{p_2}{\alpha_i}\ln\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)}{p_1 + p_2}$$

Answer

We have $U_i'(c_{is}) \equiv \alpha_i e^{-\alpha_i c_{is}}$. Plug this into the f.o.c.:

$$\frac{\pi_1 \alpha_i e^{-\alpha_i c_{i1}}}{\pi_2 \alpha_i e^{-\alpha_i c_{i2}}} = \frac{p_1}{p_2}$$

Rewrite several times in order to arrive at the required expression:

$$e^{\alpha_i(c_{i2}-c_{i1})} = \frac{p_1\pi_2}{p_2\pi_1},$$

$$\ln\left(\frac{p_1\pi_2}{p_2\pi_1}\right) = \alpha_i(c_{i2} - c_{i1}) = \frac{\alpha_i}{p_2}(\hat{Y}_i - (p_1 + p_2)c_{i1})$$

This can easily be transformed into the required expression.

Observe that c_{i1} will be greater if $\hat{Y}_i$ is greater and lower if p_1 is increased while $p_1 + p_2$ is kept constant. These are reasonable effects. The effect of α_i depends on whether the logarithm is positive or negative. But in any case risk aversion (α) counteracts (i.e., dampens) the effect of the logarithm term. If $p_1\pi_2 > p_2\pi_1$, it means that state 2 is the more attractive, the logarithm would be positive, reducing c_{i1}. A higher risk aversion would dampen this effect on c_{i1}.

4(c)

Define $\alpha \equiv 1/\left(1/\alpha_1 + 1/\alpha_2\right)$, and define Y_s (with only one subscript, without hat) as $Y_s \equiv Y_{1s} + Y_{2s}$. Show that in equilibrium the relative price is given as

$$p = \frac{p_1}{p_2} = \frac{\pi_1}{\pi_2} e^{\alpha(Y_2 - Y_1)}$$

Answer

While the two individuals regard the prices p_1, p_2 as given, we are asked to look for a competitive equilibrium. The prices are then endogenous, and there are two additional equations, supply equals demand, for both states. For state 1 this means (using the Y_1 defined in the text):

$$Y_1 = Y_{11} + Y_{21} = c_{11} + c_{21}.$$

There is a similar equation for state 2. However, by Walras' law (see any microeconomics textbook, e.g., Varian) that equation is superfluous, as it is implied by the three others, i.e., supply = demand for state 1 plus the two budget constraints. Altogether we have five equations, those three plus the two first-order conditions, to determine $p1/p2, c_{11}, c_{12}, c_{21}, c_{22}$.

The solution is found by inserting the two demand expressions and reformulating:

$$Y_1 = \frac{\hat{Y}_1 - \frac{p_2}{\alpha_1} \ln\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)}{p_1 + p_2} + \frac{\hat{Y}_2 - \frac{p_2}{\alpha_2} \ln\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)}{p_1 + p_2}$$

$$= \frac{1}{p_1 + p_2}\left[p_1 Y_{11} + p_2 Y_{12} + p_1 Y_{21} + p_2 Y_{22} - p_2 \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) \ln\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)\right]$$

Introduce now $\alpha \equiv 1/\left(1/\alpha_1 + 1/\alpha_2\right)$, a kind of aggregate measure of risk aversion, and find

$$Y_1 = \frac{1}{\frac{p_1}{p_2}+1}\left[\frac{p_1}{p_2}Y_1 + Y_2 - \frac{1}{\alpha}\ln\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)\right].$$

This implies

$$Y_2 - Y_1 = \frac{1}{\alpha}\ln\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right),$$

which implies

$$e^{\alpha(Y_2-Y_1)} = \frac{\pi_2 p_1}{\pi_1 p_2},$$

which gives the required expression for the relative price,

$$\frac{p_1}{p_2} = \frac{\pi_1}{\pi_2}e^{\alpha(Y_2-Y_1)}.$$

4(d)

Give an economic interpretation of how the equilibrium relative price depends on π_1/π_2, on $Y_2 - Y_1$, and on α_1 and α_2.

Answer

If state 1 becomes more probable, there will be a higher willingness to pay for c_{i1}, thus a higher p_1 (since no additional supply is available). If $Y_2 - Y_1$ increases, consumption in state 1 becomes more scarce relative to consumption in state 2, and p_1 will increase. The effect of a higher α magnifies the effect of $Y_2 - Y_1$ through risk aversion: The extra willingness to pay for consumption in the state with less total consumption is higher if risk aversion is high.

4(e)

Assume now $Y_1 = Y_2$. What is now the formulae for the relative price and the optimal consumptions? Give an economic interpretation of this case.

Answer

We find $\frac{p_1}{p_2} = \frac{\pi_1}{\pi_2}$ and $c_{i1} = \hat{Y}_i/(p_1 + p_2)$, and the same in state 2, $c_{i2} = \hat{Y}_i/(p_1 + p_2)$. So in this case the sum of the individuals' endowments are the same in both states, which makes it possible to achieve full insurance, i.e., both consumers can obtain risk free consumption. Indeed, this is also the market solution.

4(f)

Assume now that in addition to the future period discussed above, there is a period we may call "today." Consider two investment projects which require the same outlay today. The first gives an income x in the future period, the same irrespective of which state is realized. The second gives an income z in state 1, but nothing in state 2. What do the equilibrium prices from (c) above tell us about (i) the agents' rankings of these two projects, and (ii) the agents' willingness today to pay for the projects (i.e., what are the maximum outlays today that would lead the agents to accept the projects)?

Answer

As usual we assume that the market prices can be used to find the values of the projects, i.e., that they are small in relation to the total economy. So we use the relative price p_1/p_2 from part (c). The gross value of project 1 in the market, without deduction for the outlay, is

$$p_1 x + p_2 x = p_2 \cdot \left(\frac{p_1}{p_2} x + x\right) = p_2 x \left(\frac{p_1}{p_2} + 1\right).$$

The gross value of project 2 is

$$p_1 z + 0 = p_2 z \cdot \frac{p_1}{p_2}.$$

Project 1 is preferred if

$$x\left(\frac{p_1}{p_2} + 1\right) > z \cdot \frac{p_1}{p_2}.$$

So even if we do not know p_2, knowledge of p_1/p_2 is sufficient to decide.

However, there is no information in the market about the willingness to pay today for these projects. The outlay is the same, so we may rank them, but we cannot compare to the outlay and decide whether the net value is positive. This is because the model has no market for consumption today, only an exchange between consumption in different states in the future. This is different from p. 12 of 28 April.