

## Overview of today's lecture

- Purpose: Construct *Capital Asset Pricing Model*, CAPM
- Equilibrium model for stock market in closed economy
- First: Describe opportunity set in  $\sigma, \mu$  diagram
  - With only two risky assets
  - With many risky assets
  - With one risky and one risk free asset
  - With many risky and one risk free asset
- Then: Optimal choice based on mean-variance preferences
- Then: Consequences for equilibrium prices (more next time)

## Seminars starting next week, 29 and 30 January

- There will be two parallel seminar groups, Tuesdays 16:15–18 and Wednesdays 8:15–10, see web page
- Exercises for the first seminars will be placed on web page before noon Wednesday 23 January
- Please prepare yourselves: Try to solve the exercises!

## Mean-var opportunity set, two risky assets

Investor may construct (any) portfolio of (only) two risky assets. What is opportunity set in  $(\sigma_p, \mu_p)$  diagram?

$$W_0 = W_{10} + W_{20}$$

$$\begin{aligned} \tilde{W} &= W_{10}(1 + \tilde{r}_1) + W_{20}(1 + \tilde{r}_2) = W_0 \left[ \frac{W_{10}}{W_0}(1 + \tilde{r}_1) + \frac{W_{20}}{W_0}(1 + \tilde{r}_2) \right] \\ &= W_0[a(1 + \tilde{r}_1) + (1 - a)(1 + \tilde{r}_2)] \equiv W_0(1 + \tilde{r}_p). \end{aligned}$$

For  $j = 1, 2$ , let  $\mu_j \equiv E(\tilde{r}_j)$ ,  $\sigma_j^2 \equiv \text{var}(\tilde{r}_j)$ , and let  $\sigma_{ij} \equiv \text{cov}(\tilde{r}_1, \tilde{r}_2)$ . Then:

$$\begin{aligned} \mu_p &= a\mu_1 + (1 - a)\mu_2 \quad \left( \Rightarrow a = \frac{\mu_p - \mu_2}{\mu_1 - \mu_2} \right), \\ \sigma_p^2 &= a^2\sigma_1^2 + (1 - a)^2\sigma_2^2 + 2a(1 - a)\sigma_{12}. \end{aligned}$$

Taken together:

$$\sigma_p = \sqrt{A\mu_p^2 + B\mu_p + C},$$

where

$$\begin{aligned} A &\equiv \frac{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}{(\mu_1 - \mu_2)^2}, \\ B &\equiv \frac{-2\mu_2\sigma_1^2 - 2\mu_1\sigma_2^2 + 2\sigma_{12}(\mu_1 + \mu_2)}{(\mu_1 - \mu_2)^2}, \\ C &\equiv \frac{\mu_2^2\sigma_1^2 + \mu_1^2\sigma_2^2 - 2\mu_1\mu_2\sigma_{12}}{(\mu_1 - \mu_2)^2}. \end{aligned}$$

**Opportunity set, two risky assets, contd.**

The function  $\sigma(\mu) = \sqrt{A\mu^2 + B\mu + C}$  is called an *hyperbola*, the square root of a parabola. Both have minimum points at  $\mu = \frac{-B}{2A}$ .

**Opportunity set, two risky assets, contd.**

$$\sigma(\mu) = \sqrt{A\mu^2 + B\mu + C}$$

Asymptotes for hyperbola:

$$\mu \rightarrow \infty \Rightarrow \sigma \rightarrow \sqrt{A}\mu + \frac{B}{2\sqrt{A}},$$

$$\mu \rightarrow -\infty \Rightarrow \sigma \rightarrow -\sqrt{A}\mu - \frac{B}{2\sqrt{A}}.$$

Proof (of first part only):

$$\begin{aligned} \lim_{\mu \rightarrow \infty} [\sigma(\mu) - \sqrt{A}\mu] &= \lim_{\mu \rightarrow \infty} \frac{(\sigma(\mu) - \sqrt{A}\mu)(\sigma(\mu) + \sqrt{A}\mu)}{\sigma(\mu) + \sqrt{A}\mu} \\ &= \lim_{\mu \rightarrow \infty} \frac{(\sigma(\mu))^2 - A\mu^2}{\sigma(\mu) + \sqrt{A}\mu} = \lim_{\mu \rightarrow \infty} \frac{B\mu + C}{\sqrt{A\mu^2 + B\mu + C} + \sqrt{A}\mu} \\ &= \lim_{\mu \rightarrow \infty} \frac{B + \frac{C}{\mu}}{\sqrt{A + \frac{B}{\mu} + \frac{C}{\mu^2}} + \sqrt{A}} = \frac{B}{2\sqrt{A}}, \end{aligned}$$

and the result follows.

**Opportunity set, two risky assets, contd.**

- When  $a$  varies, the hyperbola is traced out.
- $a = 1$  gives the point  $(\sigma_1, \mu_1)$ .
- $a = 0$  gives the point  $(\sigma_2, \mu_2)$ .
- Value of  $a$  at minimum point, f.o.c.:

$$0 = \frac{d\sigma^2}{da} = 2a\sigma_1^2 - 2(1-a)\sigma_2^2 + (2-4a)\sigma_{12}$$

gives

$$a = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} \equiv a_{\min}.$$

## Short sales

- Assume it is possible to hold *negative* quantities,  $X_j < 0$ .
- Known as *short sales*.
- Buying a negative number of a security means selling it.
- Starting from nothing, selling requires *borrowing* the security first.
- Will have to hand it back in period one.
- Will also have to compensate the owner if there has been cash payouts (like dividends) in the meantime.
- In order to hand back what you borrowed and sold, you must first buy the security in the market in period one.
- Short sale raises cash in period zero, but requires outlay in period one. (Opposite of buying a security.)
- Short-seller interested in falling security prices,  $p_{j1} < p_{j0}$ .

## Minimum point of hyperbola

Is always  $a_{\min} \in [0, 1]$ ? No, will prove  $a_{\min} \in [0, \infty)$ .

Proof: Choose notation  $\sigma_1 < \sigma_2$  (so  $a$  is share of portfolio in the less risky asset). Define the *correlation coefficient*  $\rho_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2}$ . Then:

$$\begin{aligned} a_{\min} > 1 &\iff \sigma_2^2 - \sigma_{12} > \sigma_1^2 + \sigma_2^2 - 2\sigma_{12} \\ &\iff \sigma_{12} > \sigma_1^2 \iff \rho_{12} > \sigma_1/\sigma_2, \end{aligned}$$

which may or may not be true. (Only general restriction on  $\rho_{12}$ , known from statistics theory, is  $-1 \leq \rho_{12} \leq 1$ .)

Similarly:

$$a_{\min} < 0 \iff \rho_{12} > \frac{\sigma_2}{\sigma_1} > 1,$$

which is impossible.

## Hyperbola's dependence on correlation

(D&D, Appendix 6.2)

- Five constants determine shape of hyperbola:

$$\mu_1, \mu_2, \sigma_1, \sigma_2, \sigma_{12}.$$

- Assets' coordinates,  $(\sigma_1, \mu_1)$  and  $(\sigma_2, \mu_2)$ , are not sufficient.
- For fixed values of these four, let  $\sigma_{12}$  vary.
- This is *not* something the investor may choose to do, only a way to illustrate possible situations.
- Easier to discuss in terms of  $\rho_{12} \equiv \sigma_{12}/\sigma_1\sigma_2$ .
- Consider first what hyperbola looks like for  $a \in [0, 1]$ .
- Consider first the extremes,  $\rho_{12} = \pm 1$ .
- For  $\rho_{12} = 1$  and  $a \in [0, 1]$ , find  $\sigma_p = a\sigma_1 + (1 - a)\sigma_2$ .
- Linear in  $a$ , thus also in  $\mu_p$ .
- In interval between the two points: Straight line.
- Line reaches vertical axis somewhere outside interval. Kink.



**Hyperbola's dependence on correlation, contd.**

- Opposite extreme,  $\rho_{12} = -1$ , gives  $\sigma_p = \pm[a\sigma_1 - (1 - a)\sigma_2]$ .
- Also broken line, but now, kink for some  $a \in [0, 1]$ .
- Specifically, at  $a_{\min} = \frac{\sigma_2}{\sigma_1 + \sigma_2}$ .
- Summing up:
  - For  $\rho \in (-1, 1)$ , a true (strictly convex) hyperbola.
  - For extreme cases, a broken line.
  - Only for those extreme cases is  $\sigma_p = 0$  possible.
- Opportunity set consists of the hyperbola or broken line, only.
- When only two risky assets, impossible to obtain point outside (or “inside”) hyperbola.

## Mean-var portfolio choice, two risky assets

- Increasing, convex indifference curves.
- Increasing, concave opportunity set (upper half).
- Tangency point will maximize (expected) utility.
- Everyone will choose from upper half of hyperbola.
- Called *efficient set*.
- Choice within efficient set depends on preferences.
- More risk averse: Lower  $\sigma$ .

**Mean-var opportunity set:  $n$  risky assets,  $n > 2$** 

Let variance-covariance matrix of  $(\tilde{r}_1, \dots, \tilde{r}_n)$  be

$$V = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}.$$

Symmetric,  $\sigma_{n1} = \text{cov}(\tilde{r}_n, \tilde{r}_1) = \text{cov}(\tilde{r}_1, \tilde{r}_n) = \sigma_{1n}$ .

Diagonal elements were previously called  $\sigma_i^2 = \text{var}(\tilde{r}_i)$ .

If  $V$  has full rank, then impossible to construct portfolio of these assets with  $\sigma_p = 0$ . (No proof now.) Will concentrate on this case.

(If  $V$  has less than full rank, then  $\sigma_p = 0$  can be chosen. For  $n = 2$ , these were the cases  $\rho_{12} = \pm 1$ .)

Assume now  $V$  has full rank. Can be shown: Opportunity set now consists of an hyperbola and the points inside it.

Informally discussed in D&D p. 101, formally on pp. 127–132, and in the appendix of Roll's article.

**Mean-var opportunity set:  $n$  risky assets, contd.**

For any  $\mu_p$ , the agents will want as low  $\sigma_p$  as possible:

$$\min_{w_1, \dots, w_n} \sigma_p \text{ given } \mu_p$$

This defines the hyperbola called the *frontier portfolio set*.

For any  $\sigma_p$ , the agents will want as high  $\mu_p$  as possible:

$$\max_{w_1, \dots, w_n} \mu_p \text{ given } \sigma_p$$

This defines the upper half of the hyperbola. The upper half of the *frontier portfolio set* is known as the *efficient portfolio set*.

Again: *Efficient* means that part of the opportunity set from which the agents will choose, irrespective of their preferences, but within which we cannot predict their choice, since we do not specify their preferences in any more detail.

**Mean-var opport. set: One risky, one risk free asset**

- Let  $\sigma_1 = \sigma_{12} = 0$  in formulae.
- Get linear relation between  $\sigma_p$  and  $a$ .
- Thus linear relation between  $\sigma_p$  and  $\mu_p$ .
- Simplifies. Good reason for working with  $(\sigma, \mu)$ , not  $(\sigma^2, \mu)$ .
- Opportunity set broken line.
- Again, upper half is efficient.

**Mean-var oppo. set: One risk free,  $n$  risky assets,  $n > 2$** 

- Let  $r_f$  be rate of return on risk free asset.
- Risk free asset can be combined with any portfolio of risky.
- Everyone will want max  $\mu_p$  for any given  $\sigma_p$ .
- Assume  $r_f$  less than  $\mu$  at min point.
- (Will return to opposite possibility later.)
- Then: Combination of risk free asset with *tangency portfolio* is efficient. Efficient set is *linear*.

**Mean-var portfolio choice: 1 risk free,  $n$  risky assets**

- Consider now situation with many agents.
- Assume all believe in same means, variances, covariances.
- With mean-variance preferences, all want some combination of risk free asset with *same* portfolio of risky assets, tangency.
- Straight line efficient set.
- Preferences determine preferred location along line.
- Higher risk aversion: Closer to risk free asset.
- Lower risk aversion: Above tangency portfolio: Borrow money (short sell risk free asset) and invest more than  $W_0$  in tangency portfolio.
- “Two-fund spanning”: Restriction of opportunity set to  $r_f$  and tangency portfolio is just as good as original opportunity set.
- “Separation” of portfolio composition: May leave to a fund manager to make tangency portfolio available.

## Equilibrium condition

- Everyone demands same combination of risky assets.
- Necessary condition for equilibrium: This is equal to supply.
- (More complete equilibrium model later. Mossin. Roll.)
- Agent  $h$  splits  $W_0^h = W_{0f}^h + W_{0M}^h$ .
- $W_{0f}^h$  in risk free asset, possibly negative.
- $W_{0M}^h$  in tangency portfolio, strictly positive. (Why?)
- Tangency portfolio has weights  $w_{1M}, \dots, w_{nM}$ .
- Per definition  $\sum w_{jM} = 1$ .
- Total demand for  $n$  risky assets written as vector:

$$\sum_{h=1}^H \begin{bmatrix} w_{1M} W_{0M}^h \\ \vdots \\ w_{nM} W_{0M}^h \end{bmatrix} = \begin{bmatrix} w_{1M} \\ \vdots \\ w_{nM} \end{bmatrix} \sum_{h=1}^H W_{0M}^h$$

- Total has same value composition as each part.
- This must also be value composition of supply.
- Observable, “market portfolio.”
- “Portfolio” here means a vector of weights, summing to one.
- The word “portfolio” may sometimes mean some money amount invested in each of the  $n$  assets, a vector not summing to one.



## CML, market price of risk

- Everyone combines risk free asset and market portfolio.
- Line through  $(0, r_f)$  and  $(\sigma_M, \mu_M)$  called *Capital Market Line*, CML,

$$\mu_P = r_f + \frac{\mu_M - r_f}{\sigma_M} \sigma_p.$$

- Slope,  $\frac{\mu_M - r_f}{\sigma_M}$ , sometimes called *market price of risk*.
- Shows how much must be given up in expected portfolio rate of return in order to reduce standard deviation by one unit.
- All agents have MRS between  $\mu_p$  and  $\sigma_p$  equal to this.
- Will soon see: This is relevant concept for comparing whole portfolios, but not for individual assets.

## Motivating CAPM: Covariances important

- Next derive most important formula in this part of course.
- Model known as the *Capital Asset Pricing Model*. This name also used for the main formula. Formula also called the *Security Market Line* (SML).
- Shows what determines prices of individual assets.
- First motivation: Covariances important.
- Comparing alternative portfolios, when only one of them can be chosen, have assumed *variances* of rates of return are the relevant measure of risk.
- But for individual assets, which can be combined in portfolios, the relevant measure turns out to be a *covariance* with other rates of return.
- Make two simple, motivating arguments first, without reference to any equilibrium model.
- Consider making an *equally weighted* portfolio of  $n$  assets, i.e., with all  $w_j = 1/n$ . Assume that among the rates of return, one has the maximum variance,  $\sigma_{\max}^2$ . Then

$$\lim_{n \rightarrow \infty} \sigma_p^2 = \bar{\sigma}_{ij},$$

the average covariance between rates of return, and

$$\lim_{n \rightarrow \infty} \frac{\partial \sigma_p^2}{\partial w_i} = 2\bar{\sigma}_{ij}.$$

## Proof of motivating results

Observe that

$$\sigma_p^2 = \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}.$$

An equally weighted portfolio has

$$\sigma_p^2 = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} = \frac{1}{n^2} \sum_{i=1}^n \sigma_i^2 + \frac{1}{n^2} \sum_{i=1}^n \sum_{j \neq i} \sigma_{ij}.$$

Observe that the first term satisfies

$$\frac{1}{n^2} \sum_{i=1}^n \sigma_i^2 < \frac{1}{n^2} \cdot n \cdot \sigma_{\max}^2 \rightarrow 0 \quad \Leftarrow \quad n \rightarrow \infty.$$

The second term satisfies

$$\frac{1}{n^2} \sum_{i=1}^n \sum_{j \neq i} \sigma_{ij} = \frac{n^2 - n}{n^2} \bar{\sigma}_{ij} \rightarrow \bar{\sigma}_{ij} \quad \Leftarrow \quad n \rightarrow \infty,$$

which proves the first result. Observe next that for any portfolio,

$$\frac{\partial \sigma_p^2}{\partial w_i} = 2w_i \sigma_i^2 + 2 \sum_{j \neq i} w_j \sigma_{ij}.$$

Evaluated where all  $w_i = 1/n$ , this becomes

$$2 \frac{\sigma_i^2}{n} + 2 \frac{n-1}{n} \bar{\sigma}_{ij} \rightarrow 2 \bar{\sigma}_{ij} \quad \Leftarrow \quad n \rightarrow \infty.$$

## Derivation of CAPM formula

- Consider an equilibrium, everyone holds combination of risk free asset and market portfolio.
- Will derive relation between  $\mu_j, \sigma_j$  (of any asset, numbered  $j$ ) and the economy-wide variables  $r_f, \mu_M, \sigma_M$ .
- As a *thought experiment* (only), make a portfolio with a fraction  $a$  in asset  $j$  and a fraction  $1 - a$  in the market portfolio.
- (Possible, even though  $M$  already contains  $j$ .)
- For this portfolio  $p$  we have

$$\mu_p = a\mu_j + (1 - a)\mu_M, \quad \frac{\partial \mu_p}{\partial a} = \mu_j - \mu_M,$$

$$\sigma_p = \sqrt{a^2\sigma_j^2 + (1 - a)^2\sigma_M^2 + 2a(1 - a)\sigma_{jM}},$$

$$\frac{\partial \sigma_p}{\partial a} = \frac{a\sigma_j^2 - (1 - a)\sigma_M^2 + (1 - 2a)\sigma_{jM}}{\sqrt{a^2\sigma_j^2 + (1 - a)^2\sigma_M^2 + 2a(1 - a)\sigma_{jM}}},$$

$$\left. \frac{\partial \sigma_p}{\partial a} \right|_{a=0} = \frac{\sigma_{jM} - \sigma_M^2}{\sigma_M}.$$

## Illustrating the derivation

(D&D, sect. 7.2, app. 7.1, fig. 7.1)

- Small hyperbola goes through  $M$  (i.e.,  $(\sigma_M, \mu_M)$ ).
- At  $M$  it has same tangent as large hyperbola: If not, it would have to cross over large hyperbola. But that cannot happen, since large hyperbola is frontier, and  $j$  was already available when large hyperbola was formed as frontier.
- The tangent *is* the capital market line.
- Next: Use the equality of these slopes.