Derivation of zero-beta CAPM: Efficient portfolios

- Assumptions as CAPM, except r_f does not exist.
- Argument which leads to Capital Market Line is invalid.
- (No straight line through r_f, tilted up as far as possible.)
- But still: Everyone chooses on upper half of hyperbola.
- Different preferences now imply different choices along curve.
- Thus two differences from CAPM:
  - Not same portfolio of risky assets for everyone.
  - Not same MRS between σ and μ for everyone.
- Market portfolio is convex combination of agents' portfolios.
- (Convex combination: Linear combination with weights $\in [0, 1]$.)
- Corollary 5: That convex combination also on upper half.
- Market portfolio is efficient.
- Importance of this: One efficient portfolio is *observable*.
- (Under the assumptions of the model.)

Market portfolio as convex combination

Agent h has wealth W_0^h and invests in a portfolio vector

$$x^h = \begin{pmatrix} x_1^h \\ \vdots \\ x_n^h \end{pmatrix},$$

which gives money amounts

$$\begin{pmatrix} x_1^h \\ \vdots \\ x_n^h \end{pmatrix} \cdot W_0^h = \begin{pmatrix} x_1^h \cdot W_0^h \\ \vdots \\ x_n^h \cdot W_0^h \end{pmatrix}$$

invested in the n risky assets. The market portfolio in money amounts is then

$$\begin{pmatrix} x_1^h \\ \vdots \\ x_n^h \end{pmatrix} \cdot W_0^1 + \ldots + \begin{pmatrix} x_1^h \\ \vdots \\ x_n^h \end{pmatrix} \cdot W_0^H.$$

If we divide this by the total wealth of all H agents, $W_0 = \Sigma_{h=1}^H W_0^h$, we find the market portfolio expressed as relative weights,

$$\begin{pmatrix} x_1^h \\ \vdots \\ x_n^h \end{pmatrix} \cdot \frac{W_0^1}{W_0} + \ldots + \begin{pmatrix} x_1^h \\ \vdots \\ x_n^h \end{pmatrix} \cdot \frac{W_0^H}{W_0}.$$

This is, indeed, a convex combination of the individual portfolio vectors, with weights

$$\frac{W_0^1}{W_0}, \ldots, \frac{W_0^H}{W_0},$$

all between zero and unity.

Derivation of zero-beta CAPM: CAPM equation

- Derivation of CAPM equation follows previous argument.
- (See lectures of 21 and 28 January.)
- Argument works with any efficient portfolio, not only M.
- Combine one efficient pf., e, and one other asset (or pf.), j.
- Weight a in j, $1-a$ in e. Let a vary.
- Combinations form "little" hyperbola in (σ, μ) plane.
- Little hyperbola goes through (σ_e, μ_e) and (σ_j, μ_j).
- At (σ_e, μ_e), little and big hyperbola have same slope.
- At (σ_e, μ_e), little hyperbola has the slope

$$\left.\frac{d\mu}{d\sigma}\right|_{a=0} = \frac{\mu_j - \mu_e}{(\sigma_{je} - \sigma_e^2)/\sigma_e}.$$

- Corollary 3A: Slope of big hyperbola at (σ_e, μ_e) is

$$\frac{\mu_e - \mu_{z(e)}}{\sigma_e},$$

  where $z(e)$ means frontier portfolio uncorrelated with e.

- Equality of these:

$$\frac{\mu_j - \mu_e}{(\sigma_{je} - \sigma_e^2)/\sigma_e} = \frac{\mu_e - \mu_{z(e)}}{\sigma_e} \iff \mu_j = \mu_{z(e)} + (\mu_e - \mu_{z(e)})\frac{\sigma_{je}}{\sigma_e^2}.$$

Zero-beta CAPM: Conclusion, application?

- Rewritten in more explicit notation:

$$E(\tilde{r}_j) = E(\tilde{r}_{z(e)}) + \frac{\text{cov}(\tilde{r}_j, \tilde{r}_e)}{\text{var}(\tilde{r}_e)}[E(\tilde{r}_e) - E(\tilde{r}_{z(e)})].$$

- Now replace e with M, market pf.
- This gives us "zero-beta CAPM," in Roll's notation

$$r_j = r_z + \beta_j(r_m - r_z).$$

- Asset z has $\beta_z = 0$, thus z is short for zero-beta.
- Only difference from CAPM equation: r_z replaces r_f.
- If r_f exists, it is uncorrelated with everything.
- Thus the standard CAPM is special case of zero-beta CAPM.
- How to apply the zero-beta CAPM? More complicated.
- $E(\tilde{r}_{z(M)})$ is not immediately observable.
- May be estimated from time-series data if stable over time.

Estimation, the usefulness of zero covariance

$$E(\tilde{r}_j) = E(\tilde{r}_{z(e)}) + \beta_j[E(\tilde{r}_e) - E(\tilde{r}_{z(e)})],$$

or, with $E(\tilde{\varepsilon}) = 0$,

$$\tilde{r}_j = \tilde{r}_{z(e)}(1 - \beta_j) + \beta_j \tilde{r}_e + \tilde{\varepsilon}.$$

- For estimation: Natural to think of this as linear regression.
- Is it possible to estimate β_j by OLS on the equation?
- Problem: Do not have data for $\tilde{r}_{z(e)}$.
- Instead, try to omit this variable and run regression

  $$\tilde{r}_j = \alpha_j + \beta_j \tilde{r}_e + \tilde{\varepsilon}_x,$$

  where the extended error term is $\tilde{\varepsilon}_x = \tilde{\varepsilon} + \tilde{r}_{z(e)}(1 - \beta_j)$.
- This regression will give a consistent estimate of β_j if $\tilde{r}_{z(e)}$ is uncorrelated with $\tilde{r}_e$.
- Conclusion: Very useful to have the zero covariance property.

Roll's main text: Testing the CAPM?

- Test must be based on time series data.
- (Only way to observe $E(\tilde{r}_j)$, variances, covariances)
- (Relies on assumption of stability over time.)
- Data define sample means, sample variances, s. covariances.
- These sample moments define an ex post portfolio frontier.
- Results about ex ante frontier also hold for ex post frontier.
- *If* m is ex post efficient, then

  $$r_j = r_z + \beta_j(r_m - r_z)$$

  holds for ex post data. But if not, equation will not hold.
- Trying to test whether equation holds in data?
- Amounts to testing whether m is ex post efficient.
- If not: Many researchers would reject model.
- But the valid conclusion is only: Bad data for m.
- m portfolio we observed, was not ex post efficient.
- If enough data: Ex post distribution $\rightarrow$ ex ante distn.
- Traditional rejection follows if m portfolio ex ante inefficient.
- Or traditional rejection may result from too few data.

What can be tested?

- Question whether CAPM can be tested at all.
- Roll mentions four hypotheses:

(H1) Investors consider as optimal those portfolios which are mean-variance efficient.

(H2) The market portfolio is ex ante efficient.

Also: If r_f exists:

(H3) Investors can borrow and lend at a risk free rate, r_f.

(H4) $r_{z(m)} = r_f$.

- If everyone has same beliefs (about probability distributions), then (H2) follows from (H1). Identical beliefs also necessary for (H4).
- (H3) and (H4) are only relevant if r_f exists.
- Roll: Difficult to test (H1) and (H3) directly.
- Thus we are left with two testable hypotheses: (H2) and (H4).

Roll's critical comments

- Roll considers some tests from early 70's.
- In particular Fama and MacBeth (1973).
- They presented three separate hypotheses:

(C1) $E(\tilde{r}_j)$ depends linearly on β_j.

(C2) Other risk measures than β_j do not influence $E(\tilde{r}_j)$.

(C3) Risk aversion implies $E(\tilde{r}_m) > E(\tilde{r}_{z(m)})$.

- Roll points out: If m efficient, then (C1) and (C2) will hold.
- Also: (C3) will hold independently of risk aversion.
- Thus left with one hypothesis: Market portfolio is efficient.
- Points out misleading statement in Fama and MacBeth.
- "To test (C1)–(C3), need identify efficient m portfolio."
- But if identify efficient m pf., no need to test (C1)–(C3).

Problems with "approximately correct" tests

- What if data for approximate m portfolio available?
- Assume that pf. has high correlation with actual m.
- Roll: But may then reject even if model is true.
- Example: May reject (H4) even when true.
- Happens if m is on strongly concave part of hyperbola.
- Consider small deviation in approximate m.
- Still high correlation with actual m.
- But tangent's intersection with vertical axis strongly affected.

More recent empirical results

- D&D, sect. 7.9, refers to papers by Fama and French (1992, 1993).
- Test β as explanatory variable in cross section data (U.S., 1963–90).
- Find no explanatory power.
- (Data before 1963 show *some* explanatory power of β.)
- Instead the following variables are significant:

($-$) firm's size (total market value of shares)

(+) leverage (ratio of debt to total assets)

(?) earnings-to-price ratio (this period's earnings, current share price)

(+) book-to-market ratio (book value of equity $\approx$ historical cost, market value of equity $\approx$ valuation of future earnings, cf. Tobin's q)

(In parentheses: Signs of effects on average return, E/P uncertain (U-shaped?))

- Conclusion: CAPM has serious problems as empirical model.

After Fama and French

- Conclusion of Fama and French has prompted much research.
- Practitioners still use the CAPM for valuation of stocks and projects.
- But would like to have model more in line with data.
- One example: Brennan, Wang and Xia, "Estimation and test of a simple model of intertemporal capital asset pricing," *Journal of Finance,* August 2004.
- Extension of CAPM to multiperiod model, somewhat similar to CCAPM, D&D ch. 9.
- Assume these variables vary stochastically from period to period:
  - The slope of the CML
  - The real interest rate
  - The inflation rate
- Are able to "repair the CAPM" so that it fits the data better than Fama and French's model.
- Differences between high and low book-to-market stocks can be related to changes in, e.g., the slope of the CML or the interest rate.