Markets for state-contingent claims

(*Markeder for tilstandsbetingede krav*)

- Theoretically useful framework for markets under uncertainty.
- Used both in simplified versions and in general version, known as *complete markets* (*komplette markeder*) (definition later).
- Extension of standard general equilibrium and welfare theory.
- Developed by Kenneth Arrow and Gerard Debreu about 1960.
- First and second welfare theorem hold under some assumptions.
- Not very realistic.

Description of one-period uncertainty:

- A number of different *states* (*tilstander*) may occur, numbered $\theta = 1, \ldots, N$.
- Here: N is a finite number.
- Exactly one of these will be realized.
- All stochastic variables depend on this state only: As soon as the state has become known, the outcome of all stochastic variables are also known. Any stochastic variable $\tilde{X}$ can then be written as $X(\theta)$.
- "Knowing probability distributions" means knowing probabilities of each state and the outcomes of stochastic variables in each.
- When N is finite, prob. distn.s cannot be continuous.

Securities with known state-contingent outcomes

- Consider M securities (*verdipapirer*) numbered $j = 1, \ldots, M$.
- May think of as shares of stock (*aksjer*).
- Value of one unit of security j will be $p_{j\theta}$ if state θ occurs. These values are known.
- Buying numbers X_j of security j today, for $j = 1, \ldots, M$, will give total outcomes in the N states as follows:

$$\begin{bmatrix} p_{11} & \cdots & p_{M1} \\ \vdots & & \vdots \\ p_{1N} & \cdots & p_{MN} \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ \vdots \\ X_M \end{bmatrix} = \begin{bmatrix} \Sigma\, p_{j1} X_j \\ \vdots \\ \Sigma\, p_{jN} X_j \end{bmatrix}$$

- If prices today (period zero) are $p_{10}, \ldots, p_{n0}$, this portfolio costs

$$[p_{10} \cdots p_{M0}] \cdot \begin{bmatrix} X_1 \\ \vdots \\ X_M \end{bmatrix} = \sum p_{j0} X_j$$

- Observe that the vector of X's here is *not* a vector of portfolio weights. Instead each X_j the number of shares (etc.) which is bought of each security. (For a bank deposit this would be an unusual way of counting how much is invested, but think of each krone or Euro as one share.)

Constructing a chosen state-contingent vector

If we wish some specific vector of values (in the N states), can any such vector be obtained?

Suppose we wish

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}$$

Can be obtained *if there exist N securities with* **linearly independent** (*lineært uavhengige*) *price vectors,* i.e. vectors

$$\begin{bmatrix} p_{11} \\ \vdots \\ p_{1N} \end{bmatrix}, \cdots, \begin{bmatrix} p_{N1} \\ \vdots \\ p_{NN} \end{bmatrix}$$

Complete markets

Suppose N such securities exist, numbered $j = 1, \ldots, N$, where $N \leq M$. A portfolio of these may obtain the right values:

$$\begin{bmatrix} p_{11} & \cdots & p_{N1} \\ \vdots & & \vdots \\ p_{1N} & \cdots & p_{NN} \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}$$

since we may solve this equation for the portfolio composition

$$\begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{N1} \\ \vdots & & \vdots \\ p_{1N} & \cdots & p_{NN} \end{bmatrix}^{-1} \cdot \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}$$

If there are not as many as N "linearly independent securities," the system cannot be solved in general.

If N linearly independent securities exist, the securities market is called *complete.*

The solution is likely to have some negative X_j's. Thus short selling must be allowed.

Remarks on complete markets

- To get any realism in description: N must be very large.
- But then, to obtain complete markets, the number of different securities, M, must also be very large.
- Three objections to realism:
  - Knowledge of all state-contingent outcomes.
  - Large number of different securities needed.
  - Security price vectors linearly dependent.

Arrow-Debreu securities

- *Securities with the value of one money unit in one state, but zero in all other states.*
- Also called *elementary state-contingent claims,* (*elementære tilstandsbetingede krav*), or *pure securities.*
- Possibly: There exist N different A-D securities.
- If exist: Linearly independent. Thus complete markets.
- If not exist, but markets are complete: May construct A-D securities from existing securities. For any specific state θ, solve:

$$\begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{N1} \\ \vdots & & \vdots \\ p_{1N} & \cdots & p_{NN} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

with the 1 appearing as element number θ in the column vector on the right-hand side.

State prices

The *state price* for state number θ is the amount you must pay today to obtain one money unit if state θ occurs, but zero otherwise. Solve for state prices:

$$q_\theta = [p_{10} \cdots p_{N0}] \begin{bmatrix} p_{11} & \cdots & p_{N1} \\ \vdots & & \vdots \\ p_{1N} & \cdots & p_{NN} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

State prices are today's prices of A-D securities, if those exist.

Risk-free interest rate

To get one money unit available in *all* possible states, need to buy one of *each* A-D security. Like risk-free bond. Risk-free interest rate r_f is defined by

$$\frac{1}{1+r_f} = \sum_{\theta=1}^{N} q_\theta.$$

Pricing and decision making in complete markets

All you need is the state prices. If an asset has state-contingent values

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}$$

then its price today is simply

$$[q_1 \cdots q_N] \cdot \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix} = \sum_{\theta=1}^{N} q_\theta Y_\theta.$$

- Can show this must be true for all traded securities.
- For small potential projects: Also (approximately) true. Exception for large projects which change (all) equilibrium prices.
- Typical investment project: Investment outlay today, uncertain future value. Accept project if outlay less than valuation (by means of state prices) of uncertain future value.

Absence-of-arbitrage proof for pricing rule

If some asset with future value vector

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}$$

is traded for a different price than

$$[q_1 \cdots q_N] \cdot \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix},$$

then one can construct a *riskless arbitrage,* defined as

> *A set of transactions which gives us a net gain now, and with certainty no net outflow at any future date.*

A riskless arbitrage cannot exist in equilibrium when people have the same beliefs, since if it did, everyone would demand it. (Infinite demand for some securities, infinite supply of others, not equilibrium.)

Proof contd., exploiting the arbitrage

Assume that a claim to

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}$$

is traded for a price

$$p_Y < [q_1 \cdots q_N] \cdot \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}.$$

"Buy the cheaper, sell the more expensive!"

Here: Pay p_Y to get claim to Y vector, shortsell A-D securities in amounts $\{Y_1, \ldots, Y_N\}$, cash in a net amount

$$[q_1 \cdots q_N] \cdot \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix} - p_Y > 0.$$

Whichever state occurs: The Y_θ from the claim you bought is exactly enough to pay off the short sale of a number Y_θ of A-D securities for that state. Thus no net outflow (or inflow) in period one.

Similar proof when opposite inequality. In both cases: Need short sales.

Separation principle for complete markets

- As long as firm is small enough — its decisions do not affect market prices — all its owners will agree on how to decide on investment opportunities: Use state prices.
- Everyone agree, irrespective of preferences and wealth.
- Also irrespective of probability beliefs — may believe in different probabilities for the states to occur.
- Exception: All must believe that the same N states have strictly positive probabilities. (Why?)

Individual utility maximization with complete markets

Assume for simplicity that A-D securities exist. Consider individual who wants consumption today, c_0, and in each state next period, c_θ. Budget constraint:

$$W_0 = \sum_\theta q_\theta c_\theta + c_0.$$

Let $\pi_\theta \equiv \Pr(\text{state } \theta)$. Assume separable utility function

$$u(c_0) + E[U(c_\theta)].$$

We assume that $U' > 0, U'' < 0$ and similarly for the u function. (Possibly $u() \neq U()$, maybe only because of time preference. Most typical specification is that $U() \equiv \frac{1}{1+\delta} u()$ for some time discount rate δ.)

$$\max \left[u(c_0) + \sum_\theta \pi_\theta U(c_\theta) \right] \text{ s.t. } W_0 = \sum_\theta q_\theta c_\theta + c_0$$

has f.o.c.

$$\frac{\pi_\theta U'(c_\theta)}{u'(c_0)} = q_\theta \text{ for all } \theta$$

(and the budget constraint).

Remarks on first-order conditions

$$\frac{\pi_\theta U'(c_\theta)}{u'(c_0)} = q_\theta \text{ for all } \theta.$$

Taking $q_1, \ldots, q_N$ as exogenous: For any given c_0, consider how to distribute budget across states. Higher $\pi_\theta \Rightarrow$ lower $U'(c_\theta) \Rightarrow$ higher c_θ. Higher probability attracts higher consumption.

Consider now whole securities market. For simplicity consider a pure exchange economy with no productions, so that the total consumption in each future state

$$\bar{c}_\theta = \sum_{\text{individuals}} c_\theta$$

is given. Assume also everyone believes in same $\pi_1, \ldots, \pi_N$. If some π_θ increases, everyone wants own c_θ to increase. Impossible. Equilibrium restored through higher q_θ.

Assume now $\bar{c}_\theta$ increases. Generally people's $U'(c_\theta)$ will decrease. Equilibrium restored through decreasing q_θ (less scarcity).

Risk-adjusted discount rates for public projects

(Ewijk and Tang)

Central and local governments initiate lots of projects with risky future values. Risk could be in, e.g., PQ (either P or Q or both), but more generally in the citizens' willingness to pay for the output:

- State participation in oil extraction, with uncertain future oil prices and quantities.
- Building of roads, with uncertain future willingness to pay.
- Subsidies of research, with uncertain future output and willingness to pay for output.

Important issues:

- Do the authorities have means to pool risk from many projects and spread it across many individuals so that the risk premium (the risk adjustment in the discount rates) could be avoided?
- When the authorities cannot handle risk better than the market, should they price risk as in capital markets?

Influential article by Arrow and Lind (1970) (see E and T, p. 320): Yes, the government can spread risk across many, so the risk premium for each project is approximately zero.

- But based on assumption that risks are uncorrelated with aggregate consumption (which has similar role to value of market portfolio).
- Even if uncorrelated, some risks will actually be carried by few people, who will not be able to diversify.

Take systematic risk into account

- Argument in Arrow and Lind (1970) only applies to projects with a beta of zero.
- Made clear by (the same) Lind in 1982.
- When governments evaluate projects with effects spread across many people, they should at least take into account how the project contributes to the total risk of the economy, i.e., adjust for systematic risk.
- However, not clear how to define and measure this systematic risk.
- May want to take into account some limitations of the CAPM, in particular that not all assets are traded in capital markets.
- Uncertain labor income is important source of risk in future consumption budgets.
- Also many other imperfections: Credit constraints, information problems, indivisible assets (e.g., houses), etc.
- Want to measure covariance of project rates of return with future consumption.
- May want to replace future consumption with future GDP (E and T, p. 323), easier to measure.
- Dixit and Williamson (1989) (see E and T) try to estimate covariance directly from national accounts.
- May perhaps be problems with this, but will not go into those here.

Use market prices for systematic risk?

- Theoretical argument for market prices in cost-benefit analysis:
  - Want to achieve a Pareto optimal outcome.
  - Competitive market equilibrium is a Pareto optimal outcome.
  - Governments should use market prices for ordinary commodities in order to initiate projects which have a positive net willingness to pay among the citizens.
- Exceptions, market failure: Public goods, externalities, imperfect competition, etc.
- Theoretical argument could be extended to uncertainty.
- In principle, that requires complete markets for state-contingent claims.
- If markets are not complete, government can in principle achieve a Pareto improvement compared with market solution.
- In practice this is extremely complicated.
- Practical conclusion: Use similar pricing of risk as that found in capital markets.

Public risk valuation in practice

- E and T summarize this on p. 324.
- Suggest to estimate some "beta" values directly, in particular for the wage rate.
- May instead use Monte Carlo simulation if project is very complex.
- May use market data for some types of projects.
- In Norway this is discussed in public report NOU 1997:27, chapter 9.
- Suggestion to divide projects into three categories in order to be able to decentralize decisions.
- Average riskiness, less risky, "more risky."
- In "more risky" category, suggestion to imitate required expected rates of return for similar private sector projects.
- Separate justification for this: Do not want to compete with private sector on unequal terms (see also E and T on public-private partnerships, pp. 318–319).