1 Campaign contribution model: Derivation of the equilibrium tax rate

[for a detailed discussion, look at section 3.5 in Persson Tabellini (PT) text book. Here I just present the derivation of the equilibrium tax rate, the discussion of which I could not finish during my lecture on Feb 11^{th}, 2008]

[All notations are the same as in PT]

Game:

Stage 1: Candidates A and B announce their respective platforms g_A and g_B

Stage 2: Organized groups decide whether to contribute; and if a group decides to contribute, it further decides how much to contribute

Stage 3: Voters vote

Stage 4: The winner implements the policy he/she announced before the election (Full commitment to the announcement)

We are looking for a subgame perfect Nash equilibrium (SPNE) of this game.

Note that voters are identified with respect to their income, and there are three different levels of income $y^R > y^M > y^P$. As given in PT, the total contribution to candidate A and B affect the relative popularity of the two candidates in the following way: The relative average popularity of candidate B is given by

$$h\left(C_B - C_A\right) + \tilde{\delta}$$

where $\tilde{\delta}$ follows $Uniform[-\frac{1}{2\psi}, \frac{1}{2\psi}]$.

Determination of winning probabilities given two candidates' positions g_A and g_B :

If a voter in group $J \in \{P, M, R\}$ finds herself indifferent between two candidates, it must be the case that her utility from having candidate A in power is the same as her utility from having B in power. Therefore,

$$\begin{aligned} W^J\left(g_A\right) &= W^J\left(g_B\right) + h\left(C_B - C_A\right) + \tilde{\delta} \\ \tilde{\delta} &= W^J\left(g_A\right) - W^J\left(g_B\right) + h\left(C_A - C_B\right) \end{aligned}$$

Further anyone with the bias term $\tilde{\delta}$ strictly less than $W^J\left(g_A\right) - W^J\left(g_B\right) + h\left(C_A - C_B\right)$ will find A strictly preferable over B, and vice versa. Hence the vote share of A in group $J \in \{P, M, R\}$ is the proportion of voters with the bias term less than $W^J\left(g_A\right) - W^J\left(g_B\right) + h\left(C_A - C_B\right)$. This proportion is given by

$$\begin{aligned} P[\tilde{\delta} &\leq W^J\left(g_A\right) - W^J\left(g_B\right) + h\left(C_A - C_B\right)] = \frac{W^J\left(g_A\right) - W^J\left(g_B\right) + h\left(C_A - C_B\right) - \left(-\frac{1}{2\psi}\right)}{\frac{1}{\psi}} \\ &= \frac{1}{2} + \psi[W^J\left(g_A\right) - W^J\left(g_B\right) + h\left(C_A - C_B\right)]. \qquad (1) \end{aligned}$$

Since there are three groups and their proportions are given by α^P, α^M and α^R respectively, by taking the weighted sum of (1), we can calculate the candidate A's vote share (which is equivalent to the probability of winning in this case, as the share is presented as a probability between 0 and 1)

$$\begin{aligned} p_A &= \sum_J \alpha^J \left[\frac{1}{2} + \psi[W^J\left(g_A\right) - W^J\left(g_B\right) + h\left(C_A - C_B\right)]\right] \\ &= \frac{1}{2} + \psi[W\left(g_A\right) - W\left(g_B\right) + h\left(C_A - C_B\right)] \qquad (2) \end{aligned}$$

where $W\left(g_P\right) = \sum_J \alpha^J W^J\left(g_P\right)$, $P = A, B$. Similarly, one can calculate $p_B = 1 - p_A$

Determination of group contributions given two candidates' positions g_A and g_B :

If an organized group J contributes C_A^J and C_B^J to candidates A and B respectively, the expected utility, denoted V^J to its member is given by

$$\begin{aligned} V^J &= p_A W^J\left(g_A\right) + \left(1 - p_A\right) W^J\left(g_B\right) - \frac{1}{2}\left(C_A^J\right)^2 - \frac{1}{2}\left(C_B^J\right)^2 \\ &= p_A\left(W^J\left(g_A\right) - W^J\left(g_B\right)\right) - \frac{1}{2}\left(C_A^J\right)^2 - \frac{1}{2}\left(C_B^J\right)^2 + W^J\left(g_B\right) \qquad (3) \end{aligned}$$

(for a discussion on the relevance of such a cost function, see PT).

To find the optimal C_A^J, we differentiate (3) with respect to C_A^J. to get

$$\frac{dV^J}{dC_A^J} = \left(W^J\left(g_A\right) - W^J\left(g_B\right)\right)\frac{dP_A}{dC_A^J} - C_A^J$$

Replacing $\frac{dP_A}{dC_A^J}$ by $\psi h\alpha^J$ (note that $\frac{dP_A}{dC_A^J} = \psi h\frac{dC_A}{dC_A^J}$ and $\frac{dC_A}{dC_A^J} = \alpha^J$ for an organized group (see equation (3.12) in PT)), we get

$$\frac{dV^J}{dC_A^J} = \psi h\alpha^J\left(W^J\left(g_A\right) - W^J\left(g_B\right)\right) - C_A^J$$

If $\frac{dV^J}{dC_A^J}$ is positive, then group J has incentive to contribute and they would contribute to an extent to make $\frac{dV^J}{dC_A^J}$ equal to zero. Solving, we get, in such a case,

$$C_A^J = \psi h\alpha^J\left(W^J\left(g_A\right) - W^J\left(g_B\right)\right).$$

If $\frac{dV^J}{dC_A^J}$ is negative, then group J has no incentive to contribute and therefore $C_A^J = 0$. Combining these observations together, we see that

$$C_A^J = \max\left\{0, \psi h\alpha^J\left(W^J\left(g_A\right) - W^J\left(g_B\right)\right)\right\} \tag{4}$$

A similar exercise to determine C_B^J would give

$$C_B^J = -\min\left\{0, \psi h\alpha^J\left(W^J\left(g_A\right) - W^J\left(g_B\right)\right)\right\} \tag{5}$$

Depending on the sign of $\psi h\alpha^J\left(W^J\left(g_A\right) - W^J\left(g_B\right)\right)$, it is easy to see only one of C_A^J or C_B^J can be strictly positive. Hence we get the following result: An organized group, if contributing, will contribute only to one of the candidates. And, in case it contributes, the contributing amount is given by $abs\left[\psi h\alpha^J\left(W^J\left(g_A\right) - W^J\left(g_B\right)\right)\right]$.

Furthermore, from (4) and (5), we see that

$$C_A^J - C_B^J = \psi h\alpha^J\left(W^J\left(g_A\right) - W^J\left(g_B\right)\right). \tag{6}$$

Since $C_P = \sum_J O^J\alpha^J C_P^J$ for $P = A, B$, we get

$$C_A - C_B = \sum_J O^J\alpha^J\psi h\alpha^J\left(W^J\left(g_A\right) - W^J\left(g_B\right)\right) \tag{7}$$

Determining the candidates positions:

Finally we get back to stage 1 and determine the equilibrium announcements. Given g_B, by announcing some policy g_A, candidate A gets $p_A R$ where p_A depends on g_A as given in the equation (2). The first order condition gives

$$\frac{dp_A R}{dg_A} = 0, \text{ or, } \frac{dp_A}{dg_A} = 0.$$

From (2)

$$\begin{aligned}
\frac{dp_A}{dg_A} &= \frac{d}{dg_A}\left[\frac{1}{2} + \psi[W\left(g_A\right) - W\left(g_B\right) + \psi\left(C_A - C_B\right)]\right] \\
&= \psi\frac{dW\left(g_A\right)}{dg_A} + \psi h\frac{d}{dg_A}\sum_J O^J\alpha^J\psi h\alpha^J\left(W^J\left(g_A\right)\right) \\
&\quad \text{(ignoring terms involving } g_B\text{, as we assume } g_B \text{ is given)} \\
&= \psi\sum_J\alpha^J\frac{dW^J\left(g_A\right)}{dg_A} + \psi h\sum_J O^J\alpha^J\psi h\alpha^J\frac{dW^J\left(g_A\right)}{dg_A} \\
&= \psi\left[\sum_J\alpha^J\left(1 + O^J\alpha^J\psi h^2\right)\frac{dW^J\left(g_A\right)}{dg_A}\right]
\end{aligned}$$

[Recall from our discussion on Chapter 3, PT, that $W^J(g) = (y-g)\frac{y^J}{y} + H(g)$, therefore $\frac{dW^J(g_A)}{dg_A} = -\frac{y^J}{y} + H_g(g)$.]

Hence,

$$\frac{dp_A}{dg_A} = 0 \Leftrightarrow \sum_J \alpha^J \left(1 + O^J \alpha^J \psi h^2\right) \frac{dW^J(g_A)}{dg_A} = 0$$

$$\Leftrightarrow \sum_J \alpha^J \left(1 + O^J \alpha^J \psi h^2\right) \left(-\frac{y^J}{y} + H_g(g)\right) = 0$$

After rearranging terms, we get

$$\Leftrightarrow H_g(g) \sum_J \alpha^J \left(1 + O^J \alpha^J \psi h^2\right) = \sum_J \alpha^J \left(1 + O^J \alpha^J \psi h^2\right) \frac{y^J}{y}$$

$$\Leftrightarrow H_g(g) = \frac{\sum_J \alpha^J \left(1 + O^J \alpha^J \psi h^2\right) \left(y^J / y\right)}{\sum_J \alpha^J \left(1 + O^J \alpha^J \psi h^2\right)} = \frac{\hat{y}}{y}$$

$$\text{where } \hat{y} = \frac{\sum_J \alpha^J \left(1 + O^J \alpha^J \psi h^2\right) y^J}{\sum_J \alpha^J \left(1 + O^J \alpha^J \psi h^2\right)}.$$

This gives us the optimal choice of $g_A = H_g^{-1}(\hat{y}/y)$. A similar exercise to determine g_B would give you that optimal choice of g_B is also $H_g^{-1}(\hat{y}/y)$. There is an easier way to prove that both candidates would announce the same policy. If you look at the objective function of the candidate B, you would see that the function would exactly be the same as the objective function of the candidate A (after interchanging the role of g_A and g_B).

[If you find any typo in my derivation, please let me know.]